Equation Section 1

The bivariate Power-Normal and the bivariate Johnson's System bounded distribution in forestry, including height curves.

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Some print errors do occur.

Abstract

A bivariate diameter and height distribution yields a unified model of a forest stand. The bivariate Johnson's System bounded distribution and the bivariate power-normal distribution are explored. The power-normal originates from the well-known Box-Cox transformation. As evaluated by the bivariate Kolmogorov-Smirnov distance, the bivariate power-normal distribution seems to be superior to the bivariate Johnson's System bounded distribution.

The conditional median height given the diameter is a possible height curve and is compared with a simple hyperbolic height curve. Evaluated by the height deviance, the hyperbolic function yields the best height prediction. A close second is the curve generated by a bivariate power-normal distribution. Johnson's System bounded distributions suffer from the sigmoid shape of the association between height and diameter.

The bivariate power-normal is easy to estimate with good numerical properties. The bivariate power-normal is a good candidate model for use in forest stands.

Key words

Bi-normal, bivariate Johnson's System bounded distribution, bivariate power-normal distribution, height curve, Box-Cox transformation.

Introduction

A major task in forestry is to predict the distribution of diameters or heights of a forest stand. The consideration of the possible shapes of a theoretical distribution was addressed by Hafley and Schreuder (1977), who introduced the Johnson System of distributions (Johnson 1949) and the Weibull and Gamma distributions into forestry. Prediction functions that use a maximum likelihood (ML) estimation of the Johnson's System Bounded (S_B) distribution were proposed by Mønness (1982) . Some issues involved in the estimation of S_B were discussed in Lambert (1970), who introduced an improved parameterization that was revisited by Siekierski (1992) and Rennolls and Wang (2005). The Weibull distribution was reconsidered by Maltamo et al. (2000) and Merganič and Sterba (2006). A new distribution, the logit-logistic, was introduced by Wang and Rennolls (2005). The Power-Normal distribution (PN) (Freeman and Modarres 2006) originates from the well-known Box-Cox transformation to normality (Box and Cox 1964). PN was introduced into forestry by Mønness (2011b). It was shown that the PN, using only three parameters, covered a large region of the skewness x kurtosis space of shapes, comparable to S_B but also "below" the log-normal curve. The PN was found to be superior to the S_B measured by the Kolmogorov-Smirnov distribution distance in both diameter and height. On the normal scale, the PN was empirically "more normal" than the S_B. The maximum likelihood estimation of PN has better numerically properties. With S_B, numerical problems can arise in some cases, as was also reported by Siekierski (1992).

The bivariate Johnson's System bounded, described in Johnson and Kotz (1972), were introduced into forestry by Schreuder and Hafley (1977) and have been much used (Wang and Rennolls 2007). Bivariate distributions can be classified as in Johnson and Kotz (1972): 1) based on transformations of a bivariate normal and 2) based on distributions that may not be transformed into a bivariate normal. With the bi-normal assumption, the association between the two variables is the ordinary correlation coefficient on the normal scale. Estimation is performed for the two marginal distributions; the correlation is estimated as the ordinary correlation on the normal scale. Without the bi-normal assumption, the bivariate distribution is not unique, even if the marginal distributions are known. Estimation must be performed directly on the joint distribution (Wang and Rennolls 2007). They compare the bivariate Johnson's System bounded (S_{BB}), which are based on the bi-normal assumption, with what they call Plackett's Bivariate Beta, S_B and Logit-Logistic distributions. In fact, "Plackett" arises from the estimation method (Johnson and Kotz 1972). These authors conclude that "*the S_{BB} has out-performed all three bivariate Plackett-based distributional models in our empirical comparisons. "…" S_{BB} distribution is recommended*."

An alternative to using a transformed normal (or bi-normal) distribution as model for a tree distribution is to use a mixture of normal distributions (Zucchini et al. 2001). The authors report a nice fit for one stand with uneven-aged Beech. However, the comparison is against a S_{BB} with a lower bound on height close to "*minus infinity*". The authors also advocate the use of pseudo-residuals (transforming one-dimensional residuals to normal) as a measure of fit.

Predicting the height given the diameter is a common forestry task because diameter historically has been easy to measure, and height harder to measure. A large set of mathematical functions have been explored, e.g., see Fang and Bailey (1998). However, using airborne laser height measurements (Gobakken and Næsset 2004), this situation might be reversed, generating a need for a diameter-given-height relationship. A bivariate distribution is a tool for a symmetric view of diameter and height.

Methods

This article will explore the bivariate Power-Normal distribution (PN2) and the bivariate Johnson's System bounded distribution (S_{BB}). Both distributions are transformations of a bi-normal distribution, and both will also be utilized as a source for a height given diameter curve.

Consider (diameter, height) data from every tree in a stand $(D_1,H_1),\ldots,(D_n,H_n)$., with *n* being the number of trees. *D* and *H* are arbitrary diameter/height measures. *X* is either of them, whereas *Z* is a standard normal variate.

The distance between the empirical observed bi-variate cumulative distribution and the theoretical estimated bi-variate cumulative distribution will be evaluated by the Kolmogorov-Smirnov distance in two dimensions (KS2) (Peacock 1983). The distance has to be calculated at every n^2 (D_i,H_j) combinations in the plane. Unlike the one-dimensional case, where the empirical cumulative distribution function is unique, the empirical bivariate cumulative distribution can be calculated in four ways that may obtain different values: $P(D \le d \cap H \le h)$, $P(D \le d \cap H \ge h)$, $P(D \ge d \cap H \le h)$ and $P(D \ge d \cap H \ge h)$. Thus, the KS2 has four versions for each candidate distribution on each stand. Herein, all four ways are calculated, and the maximum KS2 values are reported.

An additional feature of a bivariate distribution is the median height given diameter. The curve will be estimated and compared to a simple hyperbolic function (Vestjordet 1972).

The Johnson Distributions

The Johnson's distribution system consists of three non-linear transformations of a normal variate that cover the entire skewness x kurtosis space of shapes. Johnson himself referred to these three transformations as System bounded (S_B), System lognormal (S_L), and system unbounded (S_U). Lambert (1970) introduced a statistically improved parameterization of S_B that was revisited by Rennolls and Wang (2005) and will be used herein.

Let *Z* be a standard normal variate and *X* be the observed data. S_B is defined by the non-linear transformation:

[1] $$Z = \frac{\log\left(\frac{X-\tau}{\theta-X}\right)-\mu}{\sigma}$$

where (τ,θ) are the lower and upper bounds on the *X* scale, whereas (μ,σ) are the expectation and standard deviation on the normal scale. The S_B distribution is the distribution of X . Having two S_B, that in addition are assumed bi-normal on the normal scale, yield the S_{BB} . With the bi-normal assumption, estimation can be done on the marginal distributions. ρ is the correlation on the normal scale. The S_{BB} have nine parameters $\left(\tau_D,\theta_D,\mu_D,\sigma_D,\tau_H,\theta_H,\mu_H,\sigma_H,\rho\right)$.

The Power-Normal distribution

The Power-Normal (PN) distribution originated from the Box-Cox transformation (Box and Cox 1964). Box-Cox is applicable to positive data, i.e., *X≥0.*

[2] $$Z = \frac{\frac{X^\lambda - 1}{\lambda}-\mu}{\sigma}$$ when λ≠0

[3] $$Z = \frac{\log(X)-\mu}{\sigma}$$ when λ=0

The power-normal distribution (PN) is the distribution of X . The log() case [3] is identical to the Johnson's S_L, so both PN and S_B have S_L as a limiting case. Transformation [2] is always possible, but Z can only be N(0,1) when λ=1 and when λ=0. *Z* is generally a truncated normal.

The power-normal distribution(Freeman and Modarres 2006) has the density

[4] $$f(x)=\frac{x^{\lambda-1}}{\Phi\left(sign(\lambda)k\right)\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x^\lambda-(\lambda\mu+1)}{\sigma\lambda}\right)^2} \quad , x\geq 0$$

With cumulative distribution

[5] $$F(x) = \frac{1}{\Phi(\text{sign}(\lambda)k)}\Phi\left(\frac{x^{\lambda} - (\lambda\mu + 1)}{\sigma\lambda}\right) \quad , x \geq 0$$

where $\Phi(\ )$ is the cumulative standard normal distribution function,

with $k = \frac{\lambda\mu + 1}{\sigma\lambda} = \frac{\mu}{\sigma} + \frac{1}{\sigma\lambda}$.

λ and k are shape parameters while (μ,σ) are location and scale parameters on the normal scale.

The truncation is, in forestry, in practice small because $\mu >> \sigma$; thus, the $\Phi(\text{sign}(\lambda)k) \approx 1$.

(Mønness 2011b). The parameter k was there shown to be larger than two for both diameter and height on every stand. Here, the bivariate truncation will be reported. Because the inverse distribution is almost normal, the bivariate normal distribution properties can be used to establish a bivariate PN-distribution. Thus, the association between diameter and height is the ordinary correlation on the normal scale. The PN distribution parameters, $(\lambda_D, \mu_D, \sigma_D)$ and $(\lambda_H, \mu_H, \sigma_H)$, are estimated separately. A maximum likelihood procedure for this case has been well established as the Box-Cox approach with good numerical properties[1] (Madansky 1988). The transformed data is used to estimate the usual correlation ρ . The PN2 has seven parameters $(\lambda_D, \mu_D, \sigma_D, \lambda_H, \mu_H, \sigma_H, \rho)$

Height curves

A large set of mathematical functions have been explored, e.g., see Fang and Bailey (1998) including the reference function below (Vestjordet 1972), which is also function no 12 in Fang and Bailey (1998):

[6] $h = 1.3 + \left(\frac{d}{A + Bd}\right)^2 = 1.3 + \left(Ad^{-1} + B\right)^{-2}$ named Vestjordet2 herein (“2” for second power).

[1] The truncation $\Phi(\text{sign}(\lambda)k)$ is ignored under the estimation, so it is rather an approximate maximum likelihood procedure.

where h is in meters and d is diameter at breast height (1.3 m). The function is a concave-bounded hyperbolic curve with $h(0)=1.3$ and $h(\infty)=1.3+B^{-2}$. The estimation process by a non-linear regression program (Systat Software 2012) is straightforward and numerically stable. The reason that [6] is used as a reference is that it was used in the original publication, although a simple two-parameter concave curve fits these data well. The task at hand requires no evaluation of the height curves in general.

Finding the conditional expected height given a diameter with S_{BB} and PN2 is very complicated, but because the normal is symmetric, the next best thing is to find the conditional median (or any percentile) height given a diameter.

The S_{BB} Height curve.

The conditional median height given diameter is

[7] $$h_{0.5}(d)=\tau_H+(\theta_H-\tau_H)\left(1+\left(\frac{d-\tau_D}{\theta_D-d}\right)^{-\rho\frac{\sigma_H}{\sigma_D}}e^{\left(\rho\frac{\sigma_H}{\sigma_D}\mu_D-\mu_H\right)}\right)^{-1}$$

The curve is increasing and bounded by $h_{0.5}(\tau_D)=\tau_H$ and $h_{0.5}(\theta_D)=\theta_H$ .If $\rho\frac{\sigma_H}{\sigma_D}>1$ the curve is sigmoid-shaped as seen from the d axis (Johnson and Kotz 1972). The sigmoid-shape is a drawback of the S_{BB} when applied in forestry because the H given D relation is often concave. This shape can be acceptable depending on which part of the sigmoid appears in the relevant domain of the d and h values. When $0<\rho\frac{\sigma_H}{\sigma_D}<1$, the curve is sigmoid-shaped, as seen from the h axis.

The PN2 height curve.

The conditional median height given the diameter d is given by

[8] $$h_{0.5}(d)=\left\{1+\lambda_H\mu_H-\frac{\rho\sigma_H\lambda_H}{\sigma_D\lambda_D}\left(1+\mu_D\lambda_D\right)+\frac{\rho\sigma_H\lambda_H}{\sigma_D\lambda_D}d^{\lambda_D}\right\}^{\frac{1}{\lambda_H}}$$

In a standard bi-normal, the $median\left(Z_2\middle|Z_1=z_1\right)=\rho z_1$. Substituting the *Z*'s with the Box-Cox transformation [2] yields [8]. Based on this data, λ_H and λ_D are larger than zero in every case (Mønness 2011b); thus, the curve is increasing in our setting. If $\frac{\lambda_D}{\lambda_H}<1$, the shape is concave, given that the other parameters are consistent and reasonable, and the curve is unbounded.

The statistical software SAS (SAS 2008) was used for programming; mathematical details and SAS programs are found in Mønness (2011a, a working paper). The SAS function *PROBBNRM* was used for bi-normal calculations. SYSTAT (Systat Software 2012) was used for graphics, and graphs were enhanced using a metafile program (Companion-Software 2008)

Data

The data sets were obtained from 139 young stands in South East Norway, representing both Scots Pine and Norway Spruce. The fields were established in 1954 and maintained thereafter. The diameter and height of each tree in the stands (16984 in total) were measured. The data are described in Vestjordet (1977) (In Norwegian, with a summary in English). The mean size of the plots was 420 m^2 for Scots Pine and 370 m^2 for Norway Spruce. The elevation varied from 25 to 510 m above sea level. Some stands were located on or near the coast, whereas others were located further inland. The samples were not intended to be a representative sample of young forests in southern Norway. The reasons behind the selection were as follows: 1) the mobility of the researchers was low at the time of the study, and 2) usable areas of even-aged young forest were concentrated in a few locations, because clear-cutting was not common in Norway at the time the stands were established. On the other hand, this was at the time considered a benefit, as several stands in the same area could be considered as replicates. The stands were established originally to explore the effects of pre-commercial thinning (via an early regulation of spacing, which was designed to be carried out before the stand achieved a mean height of 5 m). Both un-thinned and thinned stands are included in the data. In the thinned stands, a regime was in place under the following guidelines: 1) the remaining trees should be spaced evenly where possible; 2) the arithmetic mean heights of stands in the same area should have a small variation; 3) the height variation within a stand should be small, and the canopy should be smooth[2]; 4) deciduous trees should be removed; 5) the remaining trees should be of good quality; and 6) the mean height of a stand should be as high as possible. A summary of the stand characteristics is given in Table 1. This dataset was the empirical basis for Mønness (2011b). There, figure 2 and 7 depicted diameter and height variation. Figure 5 here depicts diameter and height relation on sample stands, including estimated height curves. The sample stands are chosen as the stands with best and worst height predictions, as measured by the deviances, see Table 4.

Results

The marginal estimation and results are documented in Mønness (2011b). The KS2 distance (maximum of four definitions) of S_{BB} and PN2 for all stands are shown in Table 2 and Figure 1. KS2=*1* is the theoretical maximum. Points (stands) on the diagonal have equal fit. PN2 is in general closer to the empirical data than S_{BB}, even if S_{BB} is close on many stands. When both distributions are far from the empirical data, S_{BB} has larger values, indicating an even lower fit. This finding is in accordance with the marginal results (Mønness 2011b). On some stands, the S_B maximum likelihood estimation encountered convergence problems. Splitting the data along this criterion did not alter the comparison (not shown).

[2] However, most stands (not shown) shows the typical concave pattern between diameter and height.

The bivariate truncation value of PN2 is also given in Table 2. The truncation, on most stands, do not appear as a significant problem assuming PN2 as bi-normal on the normal scale.

The height prediction fit was evaluated by the deviance[3], $\sqrt{\sum\left(H_{observed} - H_{predicted}\right)^2 / n}$, and by visual inspection of the *H/D* plots of every stand (not shown). Figure 2 depicts the deviance on each stand between PN2 and S_{BB} vs. Vestjordet2. Points on the diagonal have an equal fit based on the two methods in question. The curve is a LOWESS regression.

The correlations between *D* and *H* on the normal scale for both S_{BB} and PN2 are themselves highly correlated (Figure 3). The deviances (Figure 2) are also highly correlated; some stands yield good prediction by all methods and other stands yield poorer predictions by all methods. It is evident that S_{BB} has the highest deviance on most stands. Vestjordet2 and PN2 are very close. Table 3 shows the residuals from all stands, (heights are estimated within each stand), taken together. Figure 2 and Table 3 tell the same story. Vestjordet2 has the smallest mean residuals (also on nearly every stand, not shown). Often S_{BB} underestimates the heights slightly, whereas PN2 overestimates the heights slightly (Table 3, arithmetic mean).

The shape of the S_{BB} height curve is given by $\rho\frac{\sigma_H}{\sigma_D}$, whereas the shape of the PN2 height curve is given by $\frac{\lambda_D}{\lambda_H}$. These parameters are depicted in Figure 4 for each stand on a log scale. Most stands appear with $\rho\frac{\sigma_H}{\sigma_D} > 1$ and $\frac{\lambda_D}{\lambda_H} < 1$. In those cases, the S_{BB} is sigmoid-shaped, and PN2 is concave. In most other cases, the S_{BB} curve appear close to straight lines (not shown). The higher $\rho\frac{\sigma_H}{\sigma_D}$, the more

[3] The denominator has not been adjusted for the number of estimated parameters; 2 (Vestjordet2), 7 (PN2) and 9 (S_{BB}). This is for not penalizing the bivariate solutions within this comparison.

evident is the sigmoid-shape of the S_{BB}. The more evident the sigmoid-shape, the higher is the deviance of the S_{BB} height curve.

Discussion

Estimation:

First, any estimation of a bivariate distribution requires stands where both the diameter and height are measured on every tree, potentially limiting the use of this approach. Using a distribution that can be transformed into a bi-normal is superior compared to some others (Wang and Rennolls 2007). Estimation can be performed on the marginal through simple, well-known procedures. The association between diameter and height are estimated in a separate step with a common known interpretation.

A comment should be made about the PN and PN2. The calculations of exact PN moments (expectation, variance etc.) are very complicated. The Box-Cox transformation is an approximate data manipulation. The estimation of λ is based on the likelihood as if X^{λ} is normal. Else, the likelihood will be very complicated including an integral. If the truncation (=the distribution tail below $X = 0$) shows to be small, the λ is acceptable and the transformed distribution is close to normal. However, in order to use the found function as a cumulative distribution function PN or PN2, one need to ensure that $F(+\infty) = 1$, so the truncation value (which act as a scaling) must be included. This must also apply to the Kolmogorov-Smirnov statistic. The formula [8] for height given diameter assumes a bi-normal distribution, not including the truncations.

For S_B and S_{BB}, all this is no problem since the transformed data is exact normal. However, these distributions have both a lower and an upper bound, which impose other estimation issues.

Use:

With a continuous bivariate distribution, the number of trees with $(D \le d \cap H \le h)$ within a stand can be calculated from $P(D \le d \cap H \le h)$. The other intervals are found by subtraction. Confidence bands and confidence ellipsoids can be calculated based on the well-known standard normal theory.

Prediction.

Often, a prediction is based on a set of stands called the training data set, based on some regression of the parameters against the stand characteristics' "parameter prediction". This regression might not prove efficient, especially regarding shape parameters. An alternative to this path is to calculate certain stand characteristics and to perform the regression on them instead. The parameters are then recovered from the predicted characteristics "parameter recovery" See Burkhart (2012) chapter 12. S_B can be recovered from four percentiles, but the equations might not always be solvable (Mønness 1982) revisited in Siekierski (1992). PN can be recovered from three percentiles (including the median), and the equations are always solvable (Madansky 1988). Details are given in Mønness (2011a). Extending with the conditional median height given the median diameter, the correlation ρ can easily be recovered from [8].

Height given diameter prediction.

The bivariate distribution yields a unified model of distributions and the *H/D* relation. Any height given diameter percentile can be calculated, not only that of the median. This distribution is also symmetric between height and diameter.

Conclusions

The PN2 seems to be superior to the S_{BB} measured by the Kolmogorov-Smirnov statistic, both bivariate, as shown here, and marginally (Mønness 2011b).

Here, the simple Vestjordet2 height curve gave a better fit than the median height given diameter. This might be expected because Vestjordet2 exploit the height-diameter data directly, whereas PN2 and S_{BB} work indirectly through the bivariate distribution and transformed-retransformed values. However, the PN2 median height given diameter has an overall fit close to that of Vestjordet2. PN2 was proven concave on most stands. The S_{BB} has a drawback concerning the functional sigmoid-shape of the *H* given *D* association that also appears in many real cases.

The PN is easy to estimate with good numerical properties. Based on these data, the bivariate PN is a good model for studies of forest stands.

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Number of forest stands =139	Minimum	Maximum	Mean	Standard Deviation
Total age	17.0	36.0	23.6	4.2
Number of trees	33.0	527.0	122.2	61.3
No. Trees pr. hectare	1000.0	6918.9	2831.9	997.2
Dominant height at 15 year breast height, m	4.6	9.5	7.3	9.3
Dominant height *), m	4.6	14.5	8.5	20.1
Minimum tree height, m	1.2	8.5	3.0	1.5
Maximum tree height, m	5.1	15.8	9.5	2.3
Mean height, Lorey's formula Hl, m	3.7	13.3	7.4	2.0
Basal area mean diameter Dg **), cm	4.2	15.4	8.5	2.1
Minimum diameter at breast height, cm	0.3	10.7	2.5	1.8
Maximum diameter at breast height, cm	7.0	22.4	14.6	3.

*) «Dominant height» is the mean height of the 100 thickest (diameter) trees pr. hectare. Dominant Height at 15 year breast height is taken as the site index.
**) Dg is the diameter corresponding to the mean basal area of the stand.

Table 1 Stand characteristics of the 139 stands.

	Max KS2(S_{BB})	Max KS2(PN2)	Difference	Bivariate truncation value
N of Stands	139	139	139	139
Minimum	0.004	0.007	-0.021	0.984
Maximum	0.167	0.088	0.133	1.000
Median	0.027	0.021	0.007	0.999
Arithmetic Mean	0.038	0.025	0.013	0.998
Standard Deviation	0.032	0.017	0.022	0.003

Table 2 Maximum Kolmogorov-Smirnov (2 Dimensions) values of S_{BB} and PN2, and their difference. The bivariate truncation value.

	Height residuals in dm		
	PN2	S_{BB}	Vestjordet2
N of Trees	16984	16984	16984
Minimum	-31.26	-35.13	-37.20
Maximum	36.97	40.61	32.11
Arithmetic Mean	0.28	-0.25	0.05
Standard Deviation	6.54	7.44	6.47
Deviance	6.55	7.44	6.47

Table 3 Residuals around predicted heights, prediction within each stand.

Stand	PN2		S_{BB}		Vestjordet2	
	Rank	Deviance	Rank	Deviance	Rank	Deviance
13	3	2.89	20	4.58	**1**	2.90
14	**1**	2.54	**1**	2.48	3	3.06
16	113	7.72	**139**	12.10	115	7.69
55	137	10.21	135	10.76	**139**	10.19
61	**139**	10.58	136	10.83	138	10.18
123	54	5.45	56	6.33	70 (median)	5.77

Table 4. The sample stands. Ranked according to deviance value (dm). 1=best, 139=worst.

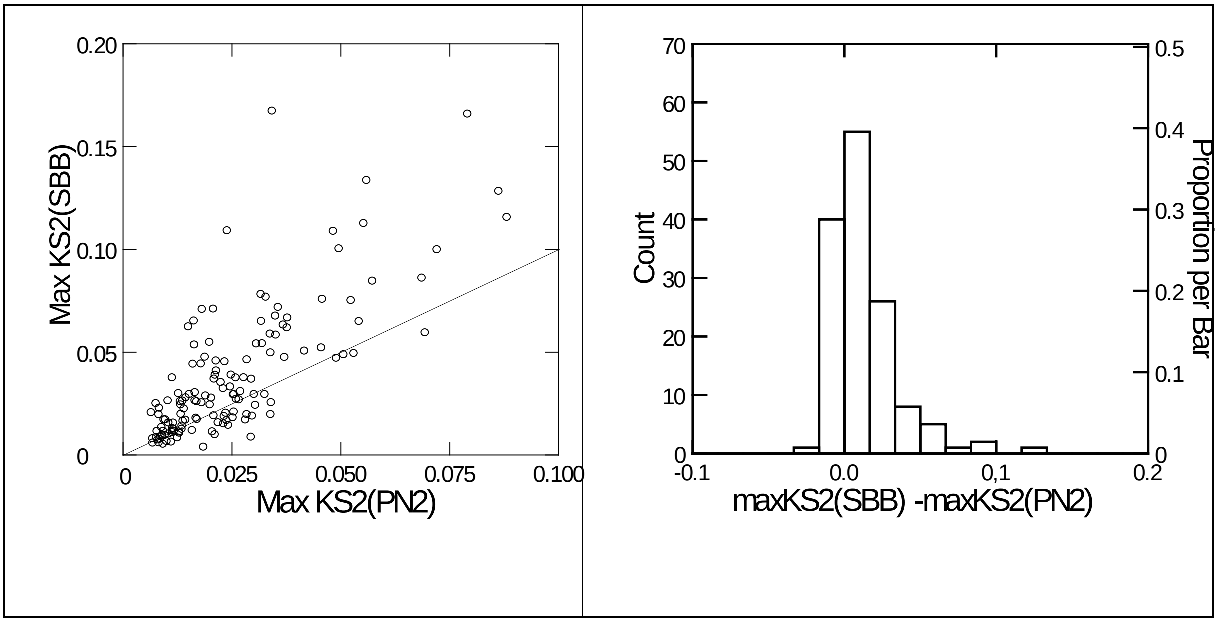


Figure 1 Maximum Kolmogorov-Smirnov (2 Dimensions) values of S_{BB} and PN2 and a histogram of their difference

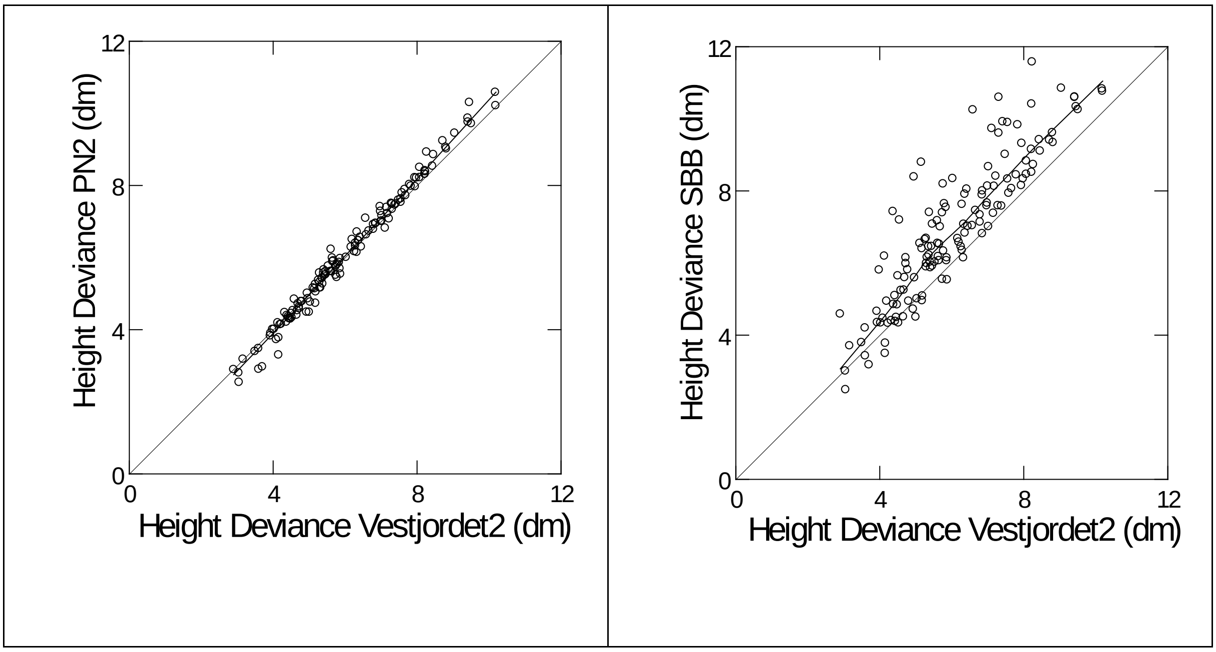


Figure 2 Deviance on each stand among the three height curve methods. The curve is a LOWESS regression.

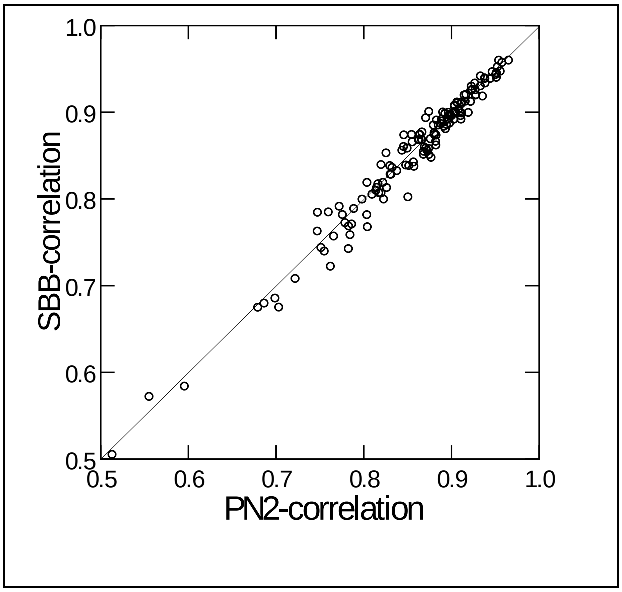


Figure 3. Correlation (D,H) on the normal scale for the S_{BB} and PN2

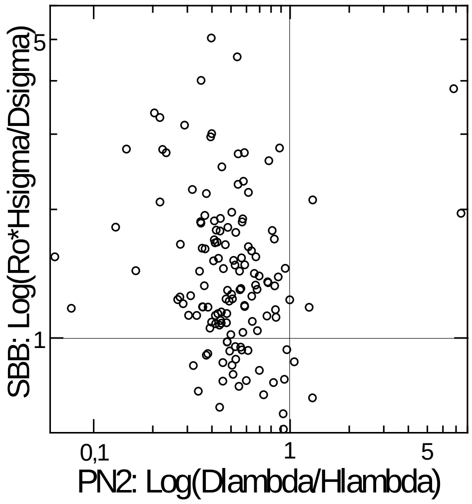
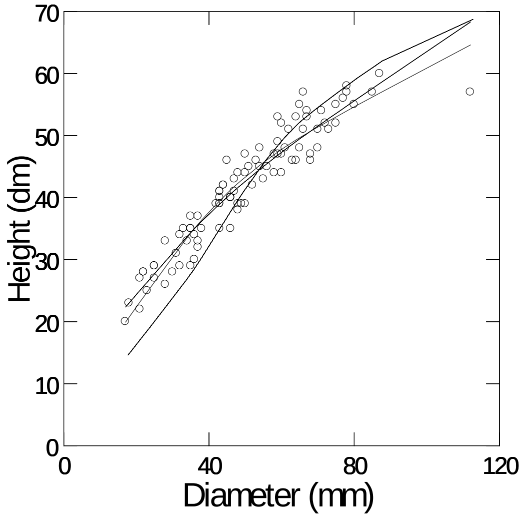


Figure 4 S_{BB} and PN2 shape parameters of the stands.

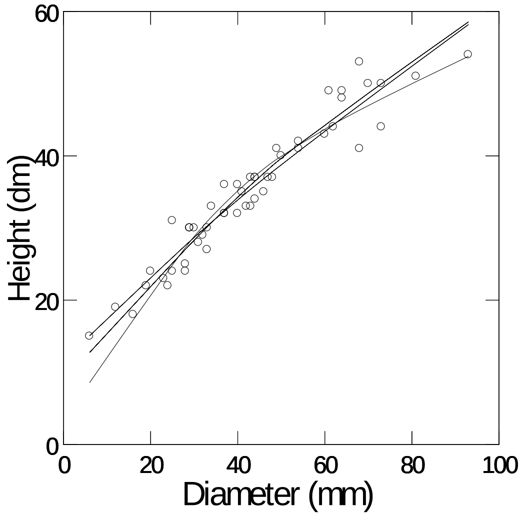
<In the published version, if possible, substitute the y-axis and x-axis text above with>

$$S_{BB}: \ \log\left(\rho \frac{\sigma_H}{\sigma_D}\right)$$

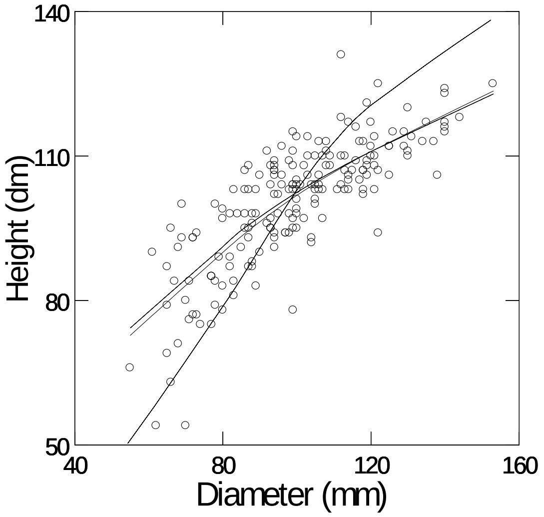
$$\mathrm{PN2}: \ \log\left(\frac{\lambda_D}{\lambda_H}\right)$$



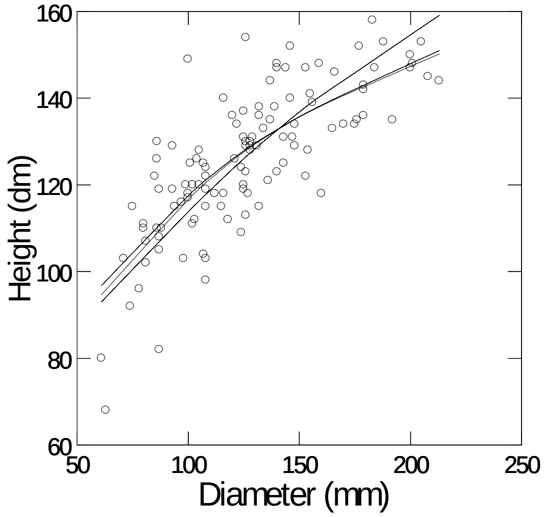
Stand 13



Stand 14



Stand 16



Stand 55

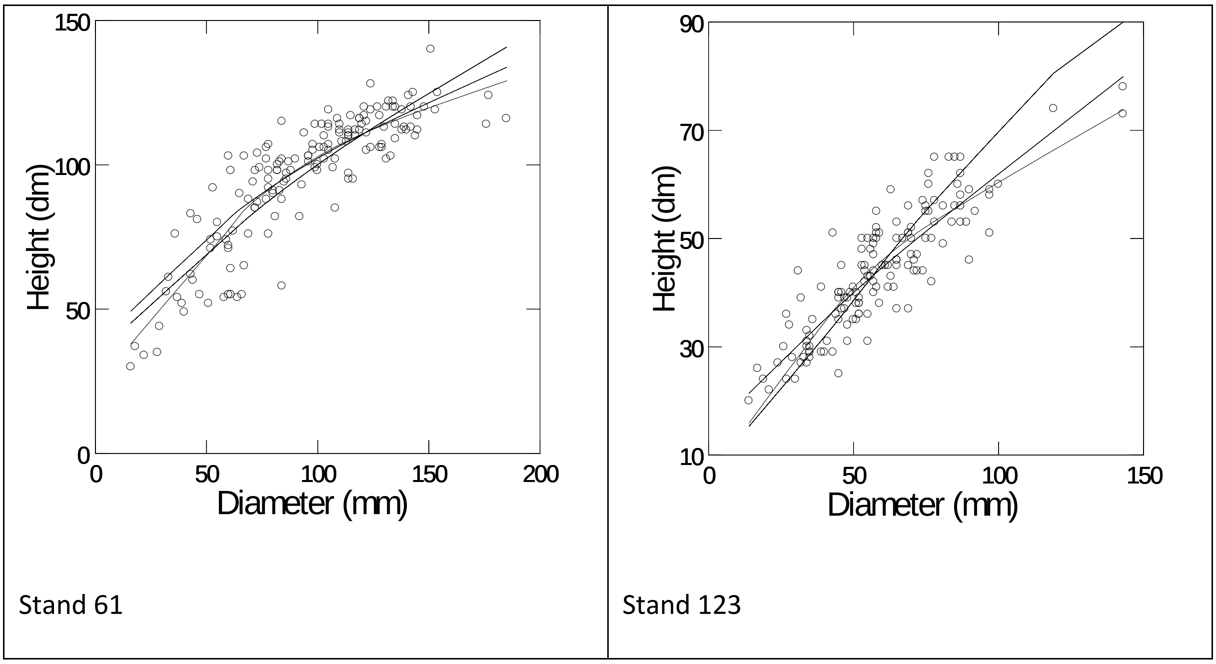


Figure 5. Sample stands depicting diameter height relation. Dotted line: S_{BB}. Dashed line: PN2. Solid line: Vestjordet2