Inland Norway
University of
Applied Sciences

Faculty of Education

Morten Bjørnebye

PhD Dissertation

Young children's grounding of mathematical thinking in sensory-motor experiences

PhD Dissertation in Teaching and Teacher Education
2021

Ph.d.-avhandlinger i profesjonsrettede lærerutdanningsfag
PhD Dissertation in Teaching and Teacher Education

Nr. 1 **Irene Trønnes Strøm** – «Jeg er ikke norsk, vet du, jeg er internasjonal». En etnografisk studie av musikalsk aktørskap blant ungdom i krysskulturelle kontekster

Nr. 2 **James Coburn** – The professional development of English language teachers. Investigating the design and impact of a national in-service EFL teacher education course

Nr. 3 **Stian Vestby** – Folkelige og distingverte fellesskap. Gentrifisering av countrykultur i Norge – en festivalstudie

Nr. 4 **Ann Margareth Gustavsen** – Kjønnsforskjeller i sosiale og skolefaglige prestasjoner

Nr. 5 **Camilla Kvaal** – Kryssende musikkopplevelser. En undersøkelse av samspill i en interkulturell praksis

Nr. 6 **Emmarentia Kirchner** – Motivating and Engaging Readers. A study of pre-adolescent Namibian primary school readers

Nr. 7 **Gro Løken** – Utsatte grupper og kjønnsforskjeller i spesialundervisning og ordinær opplæring

Nr. 8 **Elin Stengrundet** – Opprørets variasjoner. Autoritetstematikk i fire dikt av Henrik Wergeland

Nr. 9 **Even Igland Diesen** – «Når vi svinger inn på Macern gjør vi det på ordentlig!»: En studie i norsk raplyrikk

Nr. 10 **Ellen Nesset Mælan** – Læreres praksis og elevers psykiske helse: En eksplorerende studie i ungdomsskolen

Nr. 11 **Marit Elise Lyngstad** – English teachers' choices and beliefs about literature in the Norwegian upper secondary classroom

Nr. 12 **Linda Røset** – Physical education and mental health: A study of Norwegian 15-year-olds

Nr. 13 **Anne Randi Fagerlid Festøy** – Vilkår for læring for elevar på 1.-4. steg med vedtak om spesialundervisning

Nr. 14 **Inger-Kristin Larsen Vie** – Nyere, norske biografier for barn og unge: En studie av biografisjangeren og barn og unges møter med tre biografier

Nr. 15 **Siri Wieberg Klausen –** Fra kritt til programmering: En kritisk diskursanalyse av begrepet digitale ferdigheter i norsk utdanningspolitikk og i norsk videregående opplæring

Nr. 16 **Tove Sommervold** – Litterære lesemåter i ungdomsskolens litteraturundervisning. En studie av møter mellom ungdomslesere og bildebøker

Nr. 17 **Jonas Yassin Iversen** – Pre-service teachers' first encounter with multilingualism in field placement

Nr. 18 **Lars Bjørke** – The messiness and complexity of pedagogical change. Teachers' and students' experiences from implementing cooperative learning in physical education

Nr. 19 **Marianne Eek** – Språkhjelpere, investering og sosial rettferdighet. En undersøkelse av bruk av språkhjelpere i norskopplæringen for voksne nyankomne med liten skolebakgrunn

Nr. 20 **Morten Bjørnebye –** Young children's grounding of mathematical thinking in sensory-motor experiences

Morten Bjørnebye

Young children's grounding of mathematical thinking in sensory-motor experiences

PhD Thesis

2021

Faculty of Education

Inland Norway
University of
Applied Sciences

Printed by: Flisa Trykkeri A/S

Place of publication: Elverum

PhD Thesis in Teaching and Teacher Education no. 20

ISBN printed version: 978-82-8380-294-8

ISBN digital version: 978-82-8380-295-5

ISSN printed version: 2464-4390

ISSN digital version: 2464-4404

Abstract

The main objective of my dissertation is to develop knowledge about young children's grounding of mathematical thinking in sensory-motor experiences while a sub-goal is to increase the understanding of how outdoor embodied designs can facilitate such experiences. Two embodied training programmes engaging 27 children aged 3 to 5 years were conducted as a collaboration between four kindergarten teachers in two kindergartens and two researchers. These interventions led to three focal studies, where the empirical material consists of video footage of the children in individual post-tests. The data are analysed through the framework of *Embodied Cognition*, involving detailed attention to each child's cohering of task behaviour with the mathematical targeting domain addressed in the respective focal study, followed by a cross-case comparison and a multi-case analysis across and within the identified patterns of grounding of mathematical thinking in bodily action. Three different aspects were focused on: Characteristic features of subset-knowers' (i.e., children unable to use counting for exact numbering) abilities in establishing congruence between the idea of cardinality and verbalised body-spatial mapping of small sets (focal study 1); children's re-enactment of canonical structured experiences of numerosity in reasoning about additive compositions (focal study 2) and children's congruency in the physical grounding of counting based addition (focal study 3). The results showed recurring and deviating patterns of subset-knowers' grounding of the idea of cardinality in bodily production of small sets that also exceeded their knower-level, and the findings showed how sensory-motor action might concur with counting-based addition and support reasoning about additive compositions. Unexpected findings comprise the inclusion of expressive body movements (e.g., rotation, rhythm, force, and tempo) in the situating of mathematical thinking. The dissertation study contributes to the field of educational research on early structured-based bodily learning of mathematics as it revealed characteristics of young children's situating, off-loading and cohering of mathematical thinking in full-body interaction. In light of the embodied perspective, this should encourage the design of activities outdoors that involve movement and rhythm in the early learning of mathematics. In conclusion, this dissertation underlines the role that bodily movement and physical interaction with spatial structures can play in young children's mathematical thinking.

Sammendrag

Hovedmålet med avhandling min er å utvikle kunnskap om små barns forankring av matematisk tenkning i sensoriske-motoriske erfaringer, mens et delmål er å øke forståelsen for hvordan utendørs design kan støtte slike erfaringer. To intervensjoner som engasjerte 27 barn i alderen 3 til 5 år ble gjennomført som et samarbeid mellom 4 barnehagelærere i to barnehager og to forskere. Intervensjonene la grunnlaget for tre fokusstudier der det empiriske materialet består av video av barna i individuelle etter-tester. Teorien *kroppslig situert kognisjon* (*Embodied Cognition*) er brukt i analysen der en detaljert vurdering av koherens mellom oppgave adferd og det matematiske målområdet i det respektive delstudiet dannet grunnlag for sammenlikning og en fler-kasus dybdeanalyse av karakteristikker innenfor og på tvers av identifiserte mønstre for kroppslig situering av matematisk tenking. Tre forskjellige aspekter ble fokusert på: Karteristiske trekk ved delmengde-kjenneres (dvs. barn som ikke viser ferdigheter i bruk av telling for å produsere små mengder) evner til å behandle små mengder som helheter gjennom tale og kroppslig interaksjon i en stor sirkel med 50 merker (fokus studie 1); barnas evner til å gjenskape symmetrisk strukturerte kroppslige erfaringer med tallmengder for å støtte additive resonnement, og barnas evner til kroppslig modellering av tellebasert addisjon (fokus studie 3). Resultatene viste karakteristiske og divergerende trekk ved delmengde-kjenneres kroppslig situering av kardinaltallbegrepet som også omfattet produksjon av små mengder over målt begrepsnivå, og videre hvordan sensoriske-motoriske erfaringer kan støtte telle-basert addisjon og resonnement rundt del-helhet relasjoner. Avhandlingen gir et bidrag til forskningsområdet knyttet til tidlig strukturbasert kroppslig læring i matematikk, og spesielt gjennom funn som viser karakteristikker i barns situering, avlastning og koherens av matematisk tenkning i bevegelse og motorisk interaksjon. Uventede funn var inkludering av estetiske, rytmiske og sammensatte bevegelsesmønstre i den kroppsbaserte matematiske tenkningen. I lys av det kroppslige situerte perspektivet bør resultatene oppmuntre til design av utendørs aktiviteter som involverer bevegelse og rytme i den tidlige læringen av matematikk. For å konkludere understreker avhandlingen rollen som kroppslig bevegelse og fysisk interaksjon med romlige strukturer kan utgjøre i små barns matematiske tenkning.

Acknowledgements

Staten Och Kapitalet
[Ebba Grön – writer: Leif Nylén; Swedish lyrics]

Skolans uppgift är som sig bör
att skola arbetskraften
om kvastarna skall sopa bra
får man inte slarva med skaften
spärrar och kvoter och testprogram
är ett system för att sålla
agnarna från vetet och för var o en
till hans rätta fålla.

I am grateful to be delivering my dissertation after four years as a PhD candidate. This section is dedicated to all those who have enabled me to complete this inspiring and challenging process. First of all, I would like to express my appreciation to my supervisor Thorsteinn Sigurjónsson for his presence, support, encouragement, guidance and clever advice from the beginning to the end of this project. A special thanks to colleague Jorryt van Bommel for valuable help, smart advices and comments on late drafts of this document. Appreciations also to co-supervisor Reinert Rinvold, to Sevika Stensby for proofreading and to Per Steineide Refseth and Lise Iversen Kulbrandstad for valuable support and guidance. Thanks to fellow PhD candidates, dear colleagues and a special warm hug to the members of the *breakfast club*, I loved our early informal meetings before COVID-19 wreaked havoc on the world! I want to also express my gratitude to the PhD programme in *Teaching and Teacher Education* (PROFF) of *Inland University of Applied Sciences* for providing the resources, space and freedom to indulge my curiosity in exploring the epistemic potential of young children's bodily grounding of mathematical thinking. The participating children deserve every commendation as they taught me how movement in the three-dimensional room might attribute a creative, expressive, rhythmic and meaningful dimension to early learning of mathematics. I would also like to honour the kindergarten teachers who participated in the interventions and provided me with their knowledge, resources and social competencies for making the interventions a natural part of the children's everyday life. Thanks also to the leaders of the participating kindergartens for general goodwill and for allowing me to use their outdoor arenas for the implementation of the embodied designs.

Elverum, August 2021.

Table of contents

List of tables and figures

List of abbreviations

ANS	Approximate Number System
CMT	Conceptual Metaphor Theory
DBR	Design Based Research
DP	Design Principle
EC	Embodied Cognition
ECEC	Early Childhood Education and Care (institutions)
ETP	Embodied Training Programme (1 and 2)
KT	Kindergarten Teacher
MNL	Mental Number Line
OTS	Object Tracking System
SNARC	Spatial-Numerical Association of Response Codes

List of articles and conference presentations

Article 1

Bjørnebye, M., & Sigurjonsson, T. (2020). Young Children's Cross-Domain Mapping of Numerosity in Path Navigation. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics Education in the Early Years: Results from the POEM4 Conference, 2018* (pp. 109-126). Cham: Springer International Publishing.

Article 2

Bjørnebye, M., & Sigurjonsson, T. (Submitted). Young children's simulated action in additive reasoning

Article 3

Bjørnebye, M. (Under review). Full-body interaction in young children's modelling of counting-based addition

Conference presentation 1

Bjørnebye, M. (2019). Pre-schoolers' ability to synchronise multiple representations of numerosity in embodiment of a counting-on-strategy. Proceedings of CERME 11 at Utrecht.

Conference presentation 2

Bjørnebye, M. & Sigurjonsson. T. (2019). Analysing subset-knowers' cross-domain mapping of numerosity in path navigation from a perspective of Conceptual Metaphor Theory. Conference Proceedings for POEM, Kristiansand 2019.

1. Introduction: Setting the scene and outline of the dissertation

We walk. We walk. We hop. We hop.

We skip. We skip. We stop. We stop.

We jump. We jump. We bounce. We bounce.

We sit because we are done!

[Lyrics from children's exercise song, author unknown]

Full-body movement can situate inferences of numerical relations into a lived reality (Lakoff & Núñez, 2000). Cooper (1984) illustrates this by the case of physical movement in space:

> "Consider number development as learning about the space of number. In this space, one must learn where things are and how to get from one place to another. For purposes of the analogy the locations are specific numerosities and the actions to get from one place to another are additions and subtractions. How do you get from two to five? You must start in a particular direction (increasing numerosity) and go past certain landmarks (three and four) until you arrive at five (having gone a certain distance). Points in this space capture the cardinal characteristics of number: direction and landmarks, their ordinal properties; and distance, their interval properties. The developmental component in the analogy is that children learn about space of numbers by traveling in it. It is through experiences of moving in this space that children learn its ordinal structure, which is the primary content of early number development." (Cooper 1984, p. 158)

The two scenarios above suggest that mathematical ideas and thinking (e.g., ordinality, cardinality, counting, addition and subtraction) can be rooted in automated bodily behaviour, also called *Embodied Cognition* (EC). To address this potential of combining motor and cognitive processes, my qualitative dissertation study examines young children's grounding of mathematical thinking in sensory-motor experiences[1] from the perspective of EC. Framed by two *Embodied Training Programmes* (ETP 1 and ETP 2), three focal studies provide an in-depth exploration of kindergarteners' grounding of the concepts of cardinality and counting-based addition in full-body experiences and in the re-enactment of physical experiences in additive reasoning. A sub-goal is to develop knowledge about how outdoor embodied designs can support young children's mapping of mathematical ideas onto space.

I applied for a PhD position to conduct this research on early learning in mathematics for two reasons. First, after working in academia for twenty years, my experiences in creating, testing and evaluating outdoor body-based learning designs for students in early childhood education,

[1] The notion of sensory-motor experiences refers to action that requires coordination of movement and perception from the senses.

my involvement in research groups on outdoors learning and a pilot of an embodied design in kindergarten had provided me with a conviction of the benefits of using a bodily approach to modelling mathematical thinking. However, through international conferences, visiting *Early Childhood Education and Care* (ECEC) institutions and reading student evaluation of practice in kindergarten, my impression was that although much research and practice provided high quality, whole body approaches in outdoor designs to promote mathematical thinking was not properly addressed. Second, while working with students in early childhood education and care, I found it hard to find relevant and useful research and literature on a bodily approach to early learning in mathematics. The idea for this PhD emerged as a blending of these two reasons.

After I had started my PhD project, I contacted several leaders in different ECEC institutions in a municipality in eastern Norway. However, based on the argument that the KTs were too busy for a deeper engagement in the project, it was a struggle to recruit volunteers. After several setbacks and based on the assumption that the participating KTs should only be partially involved in planning, design and evaluation, I finally made an agreement with two ECEC institutions. This agreement made it possible to start my dissertation study.

The dissertation includes seven chapters. Here, in the first chapter, the context of my dissertation study is given. Chapter 2 introduces the research domain of number-space mappings in early years; it provides the rationale behind choosing to investigate the body's role in young children's mathematical thinking and using EC as a theoretical framework. It further provides arguments for choosing the outdoor as a pedagogical space for the interventions, it introduces DBR as an approach for conducting research, and it presents basic assumptions and research questions posed. Chapter 3 provides a review of literature related to the embodied numerical cognition and the mathematical target areas of the three focal studies. Chapter 4 provides a brief historical overview that foreground EC, it outlines main principles of EC, and it presents the principles for the design of the embodied training programmes. Chapter 5 presents the research methodology. This includes the choice of *Design Based Research* (DBR) and case study methodology, an outline of research design, descriptions of the embodied training programmes, information about the participating kindergartens and children, data collection methods, analytical techniques, reflections of the quality of the study and ethical considerations. Chapter 6 presents summaries of the articles. Chapter 7 discusses the findings from the focal studies in relation to the embodied framework and contemporary research on early learning of mathematics, provides reflections on the dissertation study's contributions, outlines proposals for further research and implications for practice, and presents concluding remarks and a summary.

2. Background, rationale and aims of the dissertation

2.1 Introduction to research on number-space associations in early years

Growing evidence indicates that academic achievement rests on early development of mathematical proficiency (e.g., Duncan et al., 2007; McClelland et al., 2006; Sasanguie et al., 2012). Although this recognition has led to an increased focus on high-quality mathematical learning, many early mathematical interventions focus on skill-based training at the cost of rich and playful experiences and, consequently, do not adequately match children's capacity to learn (Lewis Presser et al., 2015; van Oers, 2010). Furthermore, the international large-scale comparative PISA (*Programme for International Student Assessment*) study, whose scores and rankings have become a global gold standard of educational quality for national policy (Sjøberg, 2018), decided via their *Baby PISA programme* to include the measurement of 5-year-olds' skills in numeracy (Auld & Morris, 2019; Garvis et al., 2019). As opposed to viewing kindergarten as an academic boot camp, a growing body of research suggests a close connection between cognitive and physical development (for a review, see Valkenborghs et al., 2019). Hence, to complement existing knowledge about how social, motivational and cultural cues affect meaningful learning in early educational environments, fundamental epistemological issues concerning the connection between mathematical thinking and physical experiences must be addressed (Núñez et al., 1999).

This draws attention to the notion of *number-space mappings* (also denoted as *number-space associations* and *spatial-numerical associations*), which refers to the relation between mental magnitudes and its physical representation in space (Gallistel, 2011). Research shows that the cultivation of number-space associations begins before formal schooling (for a review, see McCrink & Opfer, 2014). The importance of early cultivation of number-space associations is underlined by empirical studies involving numerical abilities and verbal number skills (Chan & Wong, 2016; Cornu et al., 2018), exact production of small sets (Ansari et al., 2003), number sense (Bobis, 2008), addition and subtraction (Pinhas & Fischer, 2008), numerical reasoning and problem-solving (Abdullah et al., 2012). However, research on number-space associations has typically investigated children's numerical abilities based on visual stimulus (e.g., Arcavi, 2003; Gebuis & Reynvoet, 2012; Jansen et al., 2014). Support for a multimodal approach (i.e., visual, kinaesthetic, haptic-tactile, aural, verbal) comes from research suggesting that number-space associations are multifaceted, rich and flexible structures rooted in sensory and bodily experiences in the world (for a review, see Winter et al., 2015).

2.2 The turn towards ideas of embodiment in educational research in mathematics

Phenomenology, Vygotskian and semiotic-cultural, post-humanistic and ecological frameworks are examples of theoretical perspectives that contribute to insight into the body's role in learning mathematics. A phenomenological epistemology highlights haptic experiences, an ecological approach emphasises visual perception in embodied environmental interaction, while post-humanistic and semiotic-cultural perspectives underline, respectively, intra-action between body and environment (matter), and the relation between culture, body, tools and sign, as foundational for the emergence of meaning in mathematical activity (Barad, 2003; Presmeg et al., 2016; Radford et al., 2017; Roth, 2009, 2012; Streeck et al., 2011). Hence, these theoretical lenses account for various significant aspects that physical experiences can play in realising mathematical achievements, however each with its own limitations and strengths in terms of how the association between bodily action and cognitive processes (e.g., mathematical thinking) is explained. A prominent perspective that theorises both cognitive and interactionist aspects of the bodies role in mathematical thinking is EC (Stevens, 2012). This framework includes a diverse set of theories that model how mental constructs (e.g., concepts and categories) and processes (e.g., reasoning and judgement) are shaped by and grounded in sensory-motor processes (i.e., the motor and perceptual system) situated in specific environmental interaction (Borghi & Pecher, 2011).[2] Support of the embodied perspective comes from research in numerical cognition (Crollen & Noël, 2015; Jordan & Brannon, 2006; Moeller et al., 2012) and conceptual knowledge (Gallese & Lakoff, 2005), which show that sensory-motor processes are inextricably linked to thinking. The turn towards ideas of embodiment has received a growing interest in educational research that seeks to understand the body's role in achieving pedagogical goals and in particular in the learning of mathematics (e.g., Domahs et al., 2010; Goldin-Meadow et al., 2009; Pande, 2020; Shapiro & Stolz, 2019). Also, educational research has shown that whole body movements improve the retention of learned content by providing multimodal cues to represent and retrieve knowledge (Gallagher

[2] In this context, the term *embodiment* refers to the physical structure of the body (i.e., the biological body) and the experimental structure (Adenzato & Garbarini, 2012).

& Lindgren, 2015; Malinverni et al., 2014). This suggests that the adoption of the embodied perspective in educational settings is particularly important for young children who tend to support their mathematical thinking in concrete representations, bodily experiences and simulated action. Consequently, this requires a re-examination of mathematical thinking in order to design and evaluate learning environments that are consistent with the nature and function of humans conceptual systems and children's sensory-motor interaction with the world (Núñez et al., 1999). This line of reasoning concurs with the *Norwegian Framework Plan for Kindergartens*, which emphasises that the design of the physical environment in kindergartens should promote meaningful interaction and the use of the body and all senses in learning processes that build on prior motivation, knowledge and skills (Norwegian Ministry of Education and Research, 2017). However, where should embodied designs for young children in kindergarten be situated?

2.3 The outdoors as a pedaogical space in mathematics: The Norwegian context

The study by Moser and Martinsen (2010) might indicate possible locations of embodied designs, as it shows that most Norwegian kindergartens possess large outdoor areas, and that children spend more than two-thirds of their time outdoors in the summer and about one-third of their time outdoors during winter. The importance of using the outdoors as a pedagogical space is underlined by the Norwegian *Framework Plan for Kindergartens*, which emphasises that children should explore and discover mathematics in everyday life (Norwegian Ministry of Education and Research, 2017). However, the large-scale study by Reikerås et al. (2012) shows that only half of the Norwegian kindergarteners' between 30 and 33 months express number words in play, daily life activities and interplay with adults. A follow-up study of the same population shows that 4 ½ year-olds have a slower rate of development in the numerical area compared to the results found in similar research in other countries (Reikerås, 2016). Based on this, Reikerås (2016) asks whether the results mirror cultural differences suggesting that the use of mathematics may be less emphasised in communication and social interaction in Norwegian childcare contexts compared with other countries. Another influencing factor is the common conception that time outdoors should be assigned to children's free play without adult interference (Moser & Martinsen, 2010). However, mathematical acquisition of skills and concept has been described as guided reinvention (Freudenthal, 1986), and there is broad consensus that children need tutoring and help to develop their concepts and learn to pay attention to, elaborate on and mathematise aspects of everyday situations (Fuchs et al., 2007; Kirschner et al., 2006; Tudge et al., 2008). Based on the observations above, (Norwegian) children need guidance to use, discern and embody mathematical concepts in outdoor action and play, and they need designs that allow them to ground mathematical thinking in meaningful ways. This turns the attention to the conduct of educational research that includes bodily interaction in designed outdoor environments.

2.4 Design-based research from an EC perspective

In the landscape of educational studies, *Design-Based Research* (DBR) appears to be gaining increasing appeal as this perspective supports learning, creates usable and practical knowledge, and moreover, evolves theories of learning and instruction in complex settings and rich ecological environments (Prediger et al., 2015). Furthermore, the EC perspective concurs with the main objective of DBR, which is to generate empirically based theories by studying both the learning process and the supporting means and resources (DiSessa & Cobb, 2004). Based on this, principles from EC and DBR were used in the design of the two ETPs[3] included in this dissertation. Three focal studies were connected to the implementation of these programmes. The mathematical targeting areas examined in the focal studies are considered important to foster in the early years, and include exact numbering of small sets, counting-based addition and reasoning about additive compositions (e.g., De Smedt et al., 2009; Hannula et al., 2007; Nunes et al., 2012; Nunes et al., 2007; Torbeyns et al., 2015). A review of literature related to these mathematical domains follows later in the dissertation (Chapter 3). Next, I will present central assumptions that my dissertation study is based on.

2.5 The studies assumptions: Towards the aims of the dissertation study

As emphasised above, a growing body of evidence suggests that early cultivation of spatial connections to abstract concepts of numbers (cf. number-space mappings; McCrink & Opfer,

[3] In this dissertation, I chose to lean on Dackermann et al. (2017) notion of *Embodied training programme* (ETP) as "trainings that allow for an embodied experience of a specific basic numerical concept […] and that the bodily movement should specifically match the content that is trained". (p. 546). This rather instrumental definition serves only as a guideline for the design of the activities, and according to interpretation does not exclude the inclusion of cultural, personal and social factors considered salient for children's mathematical learning. See details in the outline of design principles (cf. section 4.3) and in the presentation of the activities (cf. section 5.2.1). In the text, I will sometimes use the notion of *embodied design* (Abrahamson & Lindgren, 2014) instead of ETP.

2014) is crucial to children's mathematical development. In the field of early mathematical education, there is also a recognition of the body's role in grounding of mathematical thinking in spatial extensions and locations. Yet, the direction of influence between full-body activation and mathematical thinking and its relevance to real-life activities, is still unclear (Shaki & Fischer, 2014). In particular, little is known about how children's whole-body movement and spatial interaction might concur with the logic and rules of mathematics, and this is especially the case at a more detailed level. This dissertation seeks to address these gaps, studying young children's grounding of mathematical thinking in sensory-motor experiences[1], using principles from the EC framework. The choice of an EC approach to qualitative analysis supports the identification of new ways for children to use cultural tools and ground thinking in action (Abrahamson et al., 2011), such as how they appropriate the embedded potential (affordances) that a physical modelling of mathematical thinking can bring during their action.

EC posits that thinking is not limited by the brain, but that the *body-beyond the brain* and external tools distribute, regulate and constrain internal mental processes. Consequently, EC models cognition as a product of the dynamic and bidirectional flow of information between neural and non-neural processes (Fuster, 2009). Accordingly, the study of young children's mathematical thinking from an embodied stance implies that the "the mind alone is not a meaningful unit of analysis" (Wilson, 2002, p. 626), and for this dissertation study, I need a broader unit of analyses that takes cognition, talk, interaction and artefacts into account. Based on this, I see *grounding of mathematical thinking* as number-space mappings mediated in modality-specific systems (e.g., kinaesthetic, tactile, auditory, visual-spatial; Barsalou, Simmons, et al., 2003). Consequently, the EC framework allows me to consider the coherence of mathematical thinking with talk and embodied interaction. Based on this, I conceptualise learning as a verbal and embodied process elaborated in multimodal interaction and concurrent use of multiple resources and tools (see Goodwin, 2000; Hutchins, 2006).

Furthermore, the relation between simultaneity and connection is closely associated with congruence in mathematical thinking as it allows me to study number-space mappings as both distinct phenomena (e.g., discrete motor units such as finger gestures or body parts in connection to dots) and relational phenomena (e.g., additive reasoning). Related to this, the use of gestures, body parts, physical movement and manipulation with tools to reduce cognitive demands, also known as cognitive offloading (for an overview, see Risko & Gilbert, 2016), deeper characterises how the interaction between the learner and the environment concurs with the rules of mathematical thinking. Efficiency in terms of time pressure from real-time demands, situational and contextual factors associated with physical movement is also a dimension that elaborates on how distinct number-space mappings are connected to relational thinking (Wilson, 2002; see also Caviola et al., 2017). An additional aspect concerns how previous sensory-motor experiences are re-enacted, simulated and modelled in current mathematical thinking (Hostetter & Alibali, 2019; Skoura et al., 2009). Based on these characteristics, and in light of the partial, situated, bidirectional and multimodal nature of number-space mappings (Wilson, 2002), it is important to note that congruence in mathematical thinking can be established in different constellations. This reflects the explorative dimension of the dissertation study, while the deductive dimension is based upon the logic and laws of the mathematical targeting domains addressed in the focal studies.

It is mostly the young children's grounding of thinking in embodied interaction that has been studied in the focal studies. However, few studies in early mathematics have clearly described how the designs have been enacted in practice including the guidance structures that facilitate embodied learning, the theoretical assumptions that the designs are based upon. This is especially the case for non-digital designs situated outdoors. To address this gap, a sub-goal of my dissertation study is to develop knowledge of how embodied designs can facilitate young children's grounding of mathematical thinking.

2.6 Aims and framing question

The main objective of my qualitative dissertation study is to deepen the understanding of children's grounding of mathematical thinking in embodied interaction. The framing question is:

> What characterises young children's grounding of mathematical thinking in sensory-motor experiences?

Focusing on the body's role in number-space mappings, ideas from the EC framework are used to develop knowledge of characteristics of young learners grounding of mathematical thinking in sensory-motor experiences. In this way, my study aims to contribute to educational research on early learning of mathematics, and in particular, the line of research that focuses on how whole bodily movement influences mathematical thinking. In order to delve into the framing question, I ask the following sub-questions:

(i) What characterises coherence in children's grounding of mathematical thinking in sensory-motor experiences and simulated action?

(ii) What characterises the partial, situated, bidirectional, distinct and relational nature of children's number-space mappings?

(iii) What characterises children's re-enactment and off-loading of number-space associations into gestures, body parts, bodily movement, spatial affordances and use of tools?

(iv) What characterises efficiency in children's bodily grounding of mathematical thinking?

These sub-questions address complementary aspects of the framing question (cf. section 2.5; see also section 4.2), and they are used as themes to structure the discussion in Chapter 7.

The three focal studies included in my dissertation study are reported in separate articles (referred to as *Article 1*, *2* and *3*), and are framed within the context of two outdoor embodied training programmes (referred to as ETP 1 and 2) engaging 3-to-5-year-olds' in verbalised full-body interaction. *Articles 1* and *2* were based on ETP 1 and its focus on exact production of small sets as mathematical content domain (cf. the idea of cardinality of numbers), while *Article 3* was based on ETP 2 targeting counting-based addition as the subject area. The results are analysed from an interpretative stance offering possible explanations. Focal study 1 explored congruency in 3- and 4-year-olds' grounding of the idea of cardinality in physical movement across a 50-dotted circle. Focal study 2 explored the children's abilities to support coherence in additive parts-whole reasoning in the re-enactment of the canonical (symmetric) structured embodied number-space mappings from ETP 1. Focal study 3 examined young children's ability to ground counting-based addition during physical interaction with a 100-dotted circle.

There are several arguments for claiming coherence across the focal studies. Firstly, the three mathematical targeting domains examined are building blocks of young children's learning trajectory in arithmetic (cf. Clements & Sarama, 2009; Van den Heuvel-Panhuizen, 2008). In particular, the concept of cardinality (focal study 1) plays a fundamental role in parts-whole reasoning (focal study 2) and counting based addition (focal study 3), where the two latter domains reflect different layers of the concept of addition. ETP 1's framing of both focal study 1 and 2 underscores this connection. Secondly, the focal studies reflect various aspects of young children's grounding of mathematical thinking in embodied interaction (cf. the framing question), including simulated action and the use of full- and upper-body movement to extend mathematical thinking onto space. Thirdly, EC is used as the main theoretical framework across the studies, however addressing different principles of this multifaceted framework. Finally, in order to synthesise the results of the three focal studies, the findings are discussed in relation to central assumptions of the EC-framework and contemporary research on early learning of mathematics (see Chapter 7). Combined, the different observations suggest that the three focal studies make a coherent set of investigations addressing complementary aspects of the framing question.

The dissertation's sub-goal is to develop knowledge about how outdoor embodied designs can facilitate young children's grounding of mathematical thinking, and there are two main reasons for claiming coherence to the framing question. First, the dissertation study is framed within two ETPs that concerns instruction, design, tools and learning, where the framing question address the learning aspect through the focal studies. Second, the outline of design principles (see section 4.3), the practical conduct of the programmes, rich descriptions of the activities, their relation to the design principles and issues related to re-design (see section 5.2), aim to support the transferability and credibility of my findings. However, since the empirical material in this study is mainly collected to illuminate the framing question, the epistemic processes examined in the focal studies might only indirectly point back on how the ETPs work in practice. Hence, I can only provide theoretically grounded accounts for the practical foundation (cf. the set of activities included in the embodied designs) upon which the examination of the children's mathematical thinking is based. Despite this limitation, I will discuss practical implications that the inclusion of a body-based approach to mathematical learning outdoors might entail for ECEC institutions. Hence, the aims of the DBR part of my study are to support the dissertation's scientific rigour, to derive implications of my findings to the field of practice and to generate questions for further research.

In summary, my goals with this dissertation study are to deepen the understanding of young children's grounding of mathematical thinking in sensory-motor experiences and to develop knowledge about the facilitation of such experiences in outdoor embodied designs, and thus contribute to the line of education research with an orientation towards mathematics in early childhood, including implications for such pedagogical practice. Principles from the EC framework are used to expand the knowledge regarding the grounding of mathematical thinking in embodied interaction. The empirical material of this thesis consists mainly of video recordings of individual tests post the respective interventions.

3. Review of the literature

The aim of this chapter is to position my dissertation study in the existing research literature, and the chapter consists of two parts. The first part reviews the literature on educational research connected to young children's grounding of mathematical thinking (section 3.1), while the second part (subsections 3.2-3.4) provides supplementary surveys of the literature related to the mathematical targeting areas addressed in the three articles included in my dissertation study. The literature review is also used to inform the selection of the design principles (section 4.3).

The review is the product of an ongoing and iterative search process where I used different sources and strategies. First, I made a systematic search using various combinations of terms such as *mathematics, learning, early years, numbers, number-space associations/mappings, cardinality, small sets, addition, parts-whole* and *reasoning*. Then, I combined these keywords with *embodied, grounding, whole-body, movement, gestures and design-based research*. The search involved different databases and search indexes for publishers such as Google Scholar, ScienceDirect, ERIC, JSTOR, Sage and Taylor & Francis. To complement this search strategy, I also used forward and backward snowball search (Wohlin, 2014).

3.1 Number-space mappings in educational settings

The cognitive and interactionist perspectives are two main stances of EC that I use to structure my review of research on number-space mappings in educational settings (Stevens, 2012). Although I position my dissertation study under the line of research that accounts for bodily and environmental affordances in number-space mappings, labelled *Embodied Numerical Cognition* (cf. subsection 3.1.2), I argue that the cognitive perspective presented in subsection 3.1.1 provides complementary insight into young children's body-based grounding of mathematical thinking and into the design and facilitation of such experiences.

3.1.1 The cognitive perspective of EC in educational research in mathematics

From an EC perspective on educational research, it is important to create designs that build on insight of the brain's architecture (Gallistel, 2011). Based on this, I ask how the current field of knowledge on neurocognitive research on number-space mappings in educational settings can inform the choice of DPs and the design of the ETPs in this dissertation study.

In the literature, there is consensus that cognition is grounded on at least four domain-specific core knowledge systems for representing objects, actions, number and space; each mechanism deeply rooted in human evolution (Feigenson et al., 2004; Spelke & Kinzler, 2007). Two of these foundational systems sub-serve human ability to perceive, manipulate and calculate discrete quantities (Piazza, 2010). These are: (i) The *Object Tracking System* (OTS) for rapid and precise numerical judgement of sets with one to four objects without ordinal enumeration, and (ii) the *Approximate Number System* (ANS) for rapid approximated internal analogue representations of numerical magnitudes (cf. the *Mental Number Line*; Dehaene, 2011). While the OTS is thought to underlie subitising for rapid non-verbal enumeration of small sets, the ANS is used for rapid estimation of larger sets (and possibly smaller), for comparison of non-symbolic quantities and for basic approximate arithmetic on these non-verbal mental magnitudes. The OTS and ANS are considered by many neurocognitive researchers to be main components of the notion of number sense. During early development, knowledge associated with the OTS and ANS gradually merges with verbal and symbolic representations (e.g., verbal number words, Arabic numerals) to form more coherent cognitive systems for numerical processing (Nieder & Dehaene, 2009; Torbeyns et al., 2015).
Common measures for assessing numerical magnitude understanding in terms of the ANS are non-symbolic (e.g., clouds of dots) and symbolic (i.e., Arabic numerals) versions of magnitude comparison and number line estimation tasks (Andrews & Sayers, 2015). Several studies using these measures show that early numerical magnitude understanding predicts both general mathematical performance and particular advances in subdomains such as fractions, arithmetic and algebra (e.g., Bailey et al., 2014; Reeve et al., 2012; for meta-analyses, see Schneider et al., 2017; Schneider et al., 2018).

Evidence to support the spatial nature of numerical knowledge comes from research documenting systematic spatial biases in numerical cognition, the most influencing source referred to as the *Spatial-Numerical Association of Response Codes* (SNARC) effect (Dehaene et al., 1993). The SNARC effect is closely linked to the ANS and the construct *Mental Number Line* (MNL), which models that numbers are represented spatially along a horizontal mental number line ranging from small numbers to the left and larger to the right (Dehaene, 2011). The SNARC effect shows that the response time of numbers follows the orientation of the MNL.

This means that the response time for small numbers is faster on the left side of space, and similarly, the response time is faster if the larger numbers are situated on the right side of space (for a meta-analysis, see Wood et al., 2008).

The ANS, in light of its sequential and analogue nature (cf. the MNL) is considered to support understanding of numerical rank, which is the basis for the ability to count. Research suggests that the use of the MNL is biased by cultural influences (e.g., reading direction for both words and numbers; for a review, see Göbel et al., 2011). Research also demonstrates that individuals in Western cultures tend to produce smaller random numbers if their body is turned to the left, and larger random numbers if their body is oriented to the right (Shaki & Fischer, 2014).

Educational design studies that either model structural or linear (ordinal) aspects of numbers are identified as two main approaches for enhancing mental number representations in the early years (Obersteiner et al., 2013). Support for the fostering of sequential-based knowledge of numerosities comes from studies in multi-digit addition (e.g., Ellemor-Collins & Wright, 2007) and game-based interventions aimed at reflecting the horizontal and linear nature of the MNL (e.g., Dackermann et al., 2017). For example, Ramani and Siegler (2011) found that 3- and 5-year-olds' improved numerical and arithmetic skills from playing a linear number board game, but the same authors found no improvements when the board game was circular (Siegler & Ramani, 2009). Obersteiner et al. (2013) compared the stimulation of exact and approximate enumeration for first-graders when using an exact (organised dot patterns to enhance the OTS) or approximate (random dot patterns and analogue representations to enhance the ANS) version of the same computer game, and their results showed improved abilities in the targeting domains but no crossover effects. In a related study, Wilson et al. (2009) tested the effects on kindergarteners using a computer game designed to foster number sense, and the results showed improvements in numerical comparison of digits and words (i.e., tasks traditionally used to assess number sense). However, the study showed no improvements on non-symbolic measures of number sense.

Conflicting evidence for a biased cultivation of linear aspects of numbers in educational settings comes from the Aulet and Lourenco (2018) study of spatial-numerical associations and math proficiency in 5- to 7-year-olds, which showed that children with a stronger left-to-right-oriented mental number line were less able in cross-modal arithmetic. Furthermore, the inclusion of educational designs based on patterned non-linear configurations is consistent with evidence suggesting that number-space mappings occur in three dimensions, including associations between numbers and near/far- and up/down (vertical) spaces (Winter et al., 2015).

Furthermore, most studies on number-space mappings address the MNL by the means of response times (cf. the SNARC effect; Rugani et al., 2017) and therefore do not adequately account for how numerical knowledge is experienced in real-life embodied interaction (see also criticism raised by Torbeyns et al., 2015).

Based on these observations and arguments, three relevant issues related to my dissertation emerged. Firstly, neurocognitive-based research on number-space mappings provides general support for the embodied perspective in terms of the bidirectional influence between conceptual and sensory-motor processes. Secondly, the review signifies the importance of early cultivating of spatial features of number knowledge (including the ANS and OTS). Finally, although compelling neurocognitive evidence shows that magnitude understanding predicts later abilities in mathematics, the transfer effect to educational activities is still unclear and debatable (Butterworth, 2018). In particular, most of the reviewed studies measure the response time (cf. the SNARC-effect) and/or model the linear structure of the MNL (moving objects to the left and right) in game or computer-based settings (e.g., Räsänen et al., 2009; Whyte & Bull, 2008). Therefore, these lines of investigations do not adequately integrate more culturally unbiased behaviour such as physical movement and navigation in space[4]. Furthermore, the findings from interventions designed to foster early magnitude understanding are mixed and inconclusive (Torbeyns et al., 2015). Consequently, there is a need for a better understanding of how educational designs can facilitate structured number-space associations in three dimensions in culturally realistic contexts that builds on children's motor skills in non-linear real-life situations (cf. Winter et al., 2015). To outline the window of opportunities of early body-based learning in mathematics, the next section provides a review of literature labelled under the notion of *Embodied Numerical Cognition*.

3.1.2 The interactionist perspective of EC: Embodied Numerical Cognition

Embodied Numerical Cognition is considered a prime category of EC (Bahnmueller et al., 2014), and this field of research focuses on the role of the body in number-space mappings (Dackermann et al., 2017; Moeller et al., 2012). In the research literature on *Embodied Numerical Cognition*, two complementary lines of research can be identified. One with focus

[4] Consistent with this line of reasoning is the suggestion that navigation in the three-dimensional space is, along with the OTS and ANS, considered a core inborn non-symbolic capacity of cognition (Spelke & Lee, 2012). See *Article 1* for further details.

on the effect of part-body or upper body movement on cognition and learning of number-space associations (e.g., Alibali & Nathan, 2012), and one with focus on full-body movement in the emergence and retrieval of number-space mappings. Gestures and finger counting are main categories under the first of these lines of embodied numerosity (e.g., Cook, 2011; Domahs et al., 2010), and support comes from a growing body of evidence showing that conceptual congruent gestures involving actions that match abstract mathematical ideas and relations promote performance (e.g., Segal et al., 2014). Furthermore, studies show that gestures might ease the cognitive load, provide new ideas about math, ground mathematical thought in action and bring implicit mathematical knowledge to learning (Beilock & Goldin-Meadow, 2010; Broaders et al., 2007; Cook et al., 2012; Goldin-Meadow et al., 2009). Moreover, preschool children find the use of number gestures easier than number words (Nicoladis et al., 2010). Hand movements have also been found to support young children in counting and solving numerical and arithmetic problems (Fischer & Brugger, 2011; Moeller et al., 2012).

The study by Cook et al. (2008) showed that 3rd and 4th graders who used gestures matching an addition procedure improved performance significantly better than the group that was guided to give verbal explanations. Interestingly, the results also showed a long-term effect in terms of students under the gesture condition retained more knowledge than those under the speech condition. Based on the assumption that gestural congruency in arithmetic and estimation rests on discrete and continuous actions, respectively, Segal (2011) compared 6- and 7-year-olds' performance under four conditions that reflected either congruent or incongruent simulation of arithmetic and numerical estimation. Consistent with the hypothesis, the results showed that tapping gestures outperformed sliding gestures in terms of arithmetic performance, and vice versa for numerical estimation. Furthermore, the longitudinal study by Jordan et al. (2008) showed that the use of finger gestures to solve addition and subtraction problems provides a powerful scaffolding structure in early mathematics, but that these benefits fade across age most likely due to its inefficiency to solve more complex arithmetic. In a related study, Newman (2016) found that the absence of a proper scaffolding structure in the use of fingers in early addition has a negative impact on later arithmetic performance. In sum, these variations over developmental stages shed light on both the complexity and significance of early cultivation of embodied numerosity.

The second line of research suggests a multimodal account of the notion of embodied numerosity that includes full-body spatial interaction as an integral part of the learning task (Moeller et al., 2012). This stance finds general support in studies showing that the integration of bodily activity into learning tasks can improve children's cognition, learning and academic achievements (for reviews and meta-studies, see Donnelly et al., 2016; Erickson et al., 2015; Vazou et al., 2019). Moreover, digital interfaces and technologies allow multimodal experiences through motion devices and virtual reality (Tran et al., 2017), and the integration of kinaesthetic, tactile and haptic experiences for early learning of numbers has recently been exploited in multi-touch technology (Baccaglini-Frank & Maracci, 2015).

Based on their review study, Donnelly et al. (2016) concluded that although the impact of physical activity on children's cognitive function is promising, there was limited evidence of the effects of bodily movement on learning. However sparse, some research from mathematical education reports a more pronounced training effect for young children when spatial numerical tasks were integrated with compatible bodily movement (e.g., Moeller et al., 2012). For example, Krause et al. (2019) showed sensorimotor grounding (i.e., motor force) of magnitude concept for 2.5- to 3-year-old's when they were engaged in a digital experimental task pressing a button to move objects upwards. Five additional reviewed studies are related to my dissertation study.

Firstly, the study by Beck et al. (2016) showed a better intervention effect for 7-year-olds' mathematical performance when whole body movement was integrated into the learning activity compared with the two control conditions involving sedentary fine motor skills and non-motor enriched training, respectively. However, this was not the case for low math performers, who only showed better performance in gross motor training compared to fine motor training.

Secondly, Fischer et al. (2011) found that 5-to-6-year-olds' who had received individual training in full-body movement on a digital dance mat to respond to a magnitude comparison task (step to the left/right for smaller/larger number) performed better on number line estimation compared to the group who received non-spatial numerical training. In addition, the embodied training showed a transfer effect in terms of improved flexibility in verbal counting.

Thirdly, based on the assumption of a strong relation between number line estimation and arithmetic abilities, Dackermann et al. (2016) designed an embodied training programme aimed at fostering 6- to-7-year-olds' understanding of the equidistant spacing of adjacent numbers. The one-to-one training involved walking with equally spaced steps on a large number line taped to the floor. A Kinect sensor device recorded the children's segmentation and provided video feedback about their steps. When presented with the number 3, children under the embodied condition were supposed to start at the beginning of the line and walk to the end with three equal steps (i.e., the step of one foot indicated one segment). In contrast, children under the control condition were to segment a line on a PC using an electronic pen to draw dashes. The results showed a more pronounced effect after the embodied training compared to the control condition, and the transfer effect to number line estimation and arithmetic performance was also observed to a certain extent.

Fourthly, Ruiter et al. (2015) investigated whether task-related full-body movements facilitated 7-year-olds' learning of two-digit numbers. Under two movement conditions, children were guided in representing numbers by making and simultaneously articulating different sized steps according to the value of the numerosity, making big steps for representing 10's, medium steps for 5's and small steps for 1's (e.g., two big steps, one medium step and three small steps constructed the number 28). In one of the movement conditions, they were instructed to observe their number-steps in a mirror. The results showed significantly higher test performance for the two movement conditions compared to the two non-movement conditions, and that the performance under the movement conditions was independent of mirror-based self-observation.

In the final reviewed study, Link et al. (2013) found a more profound intervention effect for first-graders' proficiency in basic numerical representations when they were engaged in full-bodily training walking the number line compared to the control condition with the same task structure without movement. It is important to note that the design aimed to reflect the continuous nature of the number line providing physical experiences of number magnitude as a walked distance from the starting point to a target number. Furthermore, the full-body responses were traced by a Microsoft Kinect device, which required the training to be conducted in a one-to-one classroom setting. This points to a limitation of several of these whole-body interventions, that the programmes are not easily transferred to real educational settings (Tran et al., 2017).

There is also evidence confirming the advantage of implementing whole body movements in the learning of other content areas. For example, Shoval (2011) found that 2nd and 3rd graders who were guided in cooperative body movement to express angles in geometric shapes performed better than the control group who received conventional teaching. However, as underlined by the reviewed literature, few studies have investigated whether and how the qualitative aspects of physical exercise may impact short and long-term cognitive performance (Pesce, 2012). In particular, there is a need for a better understanding of how full-body interaction in designed learning environments that go beyond assigning the moving body a purely instrumental role can facilitate early learning of mathematics (Malinverni et al., 2014; Ruiter et al., 2015; Shaki & Fischer, 2012).

To summarise: The review shows that an emergent line of investigations in mathematical educational research has been reflected in ideas of embodiment, that cognition is constrained and bound to the body and its actual or possible interaction with the environment (Anderson, 2003; Shapiro, 2011). Though promising, much remains unknown about how the physical body and its intrinsic dynamics might contribute to and explain mathematical learning (Pexman, 2019). However, two pertinent issues related to my dissertation study emerged from the review. Firstly, many of the reviewed interventions modelled the linear structure of the MNL (*Mental Number Line*), which stands in contrast to how young children experience numbers during physical interaction in the world. Secondly, the focus on digital technology in one-to-one settings in the classroom limits the transferability to practice. Put together, this suggests that alternative learning areas suitable for joint non-digital activities and physical movement for simulating non-linear representations of numbers should be included in the emerging line of educational research involving whole-body movement.

3.2 Children's understanding of cardinality of numbers and how it can be measured

This section reviews literature related to the mathematical target domain in focal study 1 (presented in *Article 1*), which is the concept of cardinality of numbers. First, in subsection 3.2.1, I review research emphasising on language skills in children's understanding of cardinality, and in subsection 3.2.2, I outline the measures used in this dissertation study to assess children's understanding of cardinality. Then, in subsection 3.2.3, I survey the literature on a multimodal approach to children's exact production of small sets. In subsection 3.2.4, I review the literature on structured approaches to the idea of cardinality, with a focus on research related to the notion of subitizing. In the final part (subsection 3.2.5), I summarise the review and I relate the survey to my dissertation study.

3.2.1 Early development of cardinality of numbers: A focus on the verbal modality

Cardinal knowledge of numbers involves understanding quantity as a property of sets (i.e., the wholeness of items). Findings in education research suggest that the individual's concept of cardinality in terms of fluency in exact enumeration of sets is a necessary foundation for the development of powerful arithmetic skills (Björklund, Marton, et al., 2021), and it constitutes a predictive factor for later mathematical achievements (Aunio & Niemivirta, 2010; Fischer et al., 2008). Furthermore, the ability to map between different non-symbolic quantities (e.g., arrays of dots, sets of objects) and spoken number words is foundational for understanding the symbolic number system (Purpura et al., 2013).

An understanding of the idea of cardinality as a linguistic principle is based on the integration of several sub-components (cf. the counting principles of Gelman & Gallistel, 1978), each of which takes considerable time and effort to master. A typical learning trajectory begins with early attempts at counting before 2 years of age and gradually learning the cardinal meaning of one, two and three through subitising (see subsection 3.2.4), until finally realising the relation between counting and cardinality at roughly 46 months (Le Corre & Carey, 2007; Levine et al., 2010). But the view that understanding of cardinality is measured as a linguistic principle through the question "How-many?" and thus rests on the ability to recite the number-word list and couple the last number-word to the quantity of the set, is debatable. For example, several studies show that, regardless of whether the child initiates the counting from the number word *one* or *three*, it immediately repeats the last articulated number word after being asked how many objects (e.g., Bermejo et al., 2004). This is described by Fuson (1988) as the last-word rule, a mechanically learned response of the last number-word stated in a counting sequence. In a related study, Mix et al. (2012) investigated whether different conditions with specific input helped 3 ½-year-olds to grasp the idea of the cardinal word principle. Among the training approaches (i.e., comparison condition, counting condition, naming condition, alternating condition) and the control condition, the only method that made significant improvement was to label a set's cardinality and then immediately count it (i.e., the comparison condition). The authors concluded that their results have direct educational applications, especially for children with immature numeracy skills, as the *label + count* training demonstrated a significant improvement of this core aspect of the idea of exact numbers (Mix et al., 2012). Partly conflicting evidence to the results of Mix et al. (2012) comes from Paliwal and Baroody's (2018) intervention study on 3- to 5-year-olds, which showed that both the count-first and label-first groups outperformed the counting-only group on the CP task at a delayed post-test, with the count-first condition outperforming the two other conditions. In another related study, Bermejo et al. (2004) compared a learning programme centring on cognitive conflicts with a control condition for 4- to 6-year-olds who were purposely selected according to the criteria reflected in mechanically responding to any cardinal question by repeating the last-word stated. The results showed that the children in the experimental group reached full understanding of cardinality, which means that they were able to provide accurate cardinality responses (cf. the highest level in Bermejo, 1996 six-staged model of cardinal understanding). In another study, Levine et al. (2010) found that parents' use of number words in communication with their toddlers (aged between 14 and 30 months) predicted their knowledge of the cardinal meanings of numbers words at preschool ages (46 months). Related to this, Li and Baroody (2014) reported that young children's spontaneous attention to exact quantity on a non-verbal matching task correlates with their verbal quantification skills. In another related study, Rodríguez et al. (2018) investigated the impact of iconic representations of quantity and spoken number words on the performance of 3- and 4-year-olds' when building collections of 1-6 items. The findings show that iconic representations supported the children to produce concrete sets with cardinal values that exceeded their capacities to use number words in similar mapping tasks. In a related study, Lira et al. (2017) investigated pre-schoolers abilities in mapping among different symbolic (digits and number words) and non-symbolic representations of exact quantity, and they found that mapping between written digits and non-symbolic exact quantities emerges later than the other mapping.

3.2.2 Knower-level theory and the Give-N task

A developmental model that reflects children's concepts of numbers is the *knower-level theory* (Lee & Sarnecka, 2010; Sarnecka & Carey, 2006). According to this theory, children who are unable to assign any semantic expression to the referential set are referred to as *pre-number-knowers*. Over time, they progress in number-knower level as they learn to map new number words onto the exact cardinal meaning, first *one* (*C1-knower*), then *two* (*C2-knower*), then *three* (*C3-knower*) and then *four* (*C4-knower*), before they infer how they can use counting to produce any requested set and become *Cardinal Principle knower* (*CP-knower*). Accordingly, the learning of the first four number words are learned gradually, one at a time, while the use of the cardinal principle to assign the meaning of all number words higher than *four* are learned at once, by induction (Carey, 2004). Therefore, the transition from being a *subset-knower* (i.e., C1- to C4-knower) to becoming a CP-knower marks a significant leap in the development of young children's concepts of positive integers.

The *Give-N task* is a recognised procedure for assessing children's knower-level (Schaeffer et al., 1974; Wynn, 1990, 1992), and is as follows: Contextualised with a toy figure (e.g., a puppy) and a collection of items, the child is asked to produce a certain number of objects (e.g., "Can you give the puppy two apples?"). Based on behaviour data for stable production of exact numbered sets on similar number-word mappings onto concrete representations, the cardinal-knower-level is categorised as either *subset-knower* or *CP-knower*. The Give-N procedure applies the titration method, which means that if the respondent does not know a number (they are asked in ascending order 1, 2, 3 etc.), then no larger number is asked. This means that a child whose behavioural data suggests C2-knower level will not be given the opportunity to give four items after failing in the production on the C3-level. The knower-level theory makes a series of additional assumptions of children's behaviour on the Give-N task. For example, it assumes that children will avoid producing any set size of known cardinal meaning (Wynn, 1990, 1992). This means that the children's guesses of unknown number words are lower bound by their cardinal knower level (Lee & Sarnecka, 2010). Combined with the strong predictions about the sequential progress of knower-level, this suggests that the Give-N data are diagnostic when it comes to assessing children's development of number concepts.

3.2.3 The notion of equinumerosity and research on cross-modal mapping of numbers

The term *equinumerosity* (exact equality), which originates from Piaget's (1952) conservation-of-quantity task, reflects the idea that two sets have the same cardinal value, and consequently follows the principle of one-to-one correspondence (cf. Gelman & Gallistel, 1978). Based on questions such as "Are there the same number of bricks and pencils?", Piaget (1952) found that children develop this ability at age 5 or 6. In their study of 51 children (mean age 3 years 4 months; CP-knowers, n = 22), Sarnecka and Wright (2013) tested whether this proficiency develops at earlier ages, at least for numbers five and six, given that the question is asked with particular numbers such as "There are five bricks. Are there five pencils, or six?" The results show that CP-knowers, and not subset-knowers, possess the idea of equinumerosity. Based on this, Sarnecka and Wright (2013) argue that equinumerosity together with the understanding of cardinality and the successor rule (i.e., adding one item to a set increases the cardinal number by one) should be included in an operational definition of number knowledge, or what they refer to as the idea of *exact numbers*. The authors conclude that children at an early age must understand all these three aspects of the idea of exact numbers, at least up to 10.

The notion of equinumerosity reflects that understanding of cardinality extends beyond verbal skills, which turns the attention to the literature of cross-modal mappings of numbers where the following three reviewed studies are relevant to my dissertation study. First, the case study by Sedaghatjou and Campbell (2017) used an embodied phenomenological framework for an in-depth analysis of how multimodal feedback in the form of touch, vision and auditory via a touchscreen device on an iPad using the app *TouchCounts* fostered a 4-year-old's understanding of cardinality. The authors emphasise that the multimodal approach helped the child to experience the relation between simultaneous and sequential ways of representing cardinality (Sedaghatjou & Campbell, 2017). Second, Gordon et al. (2019) explored 343 pre-schoolers' (M_{age} = 4.07 years) use of gestures under the Give-N-task. The results show that children's spontaneous use of gestures increased as a function of trial difficulty, suggesting that children map their ideas of cardinality onto gestures (i.e., kinaesthetic and visual modality) to scaffold cognitively difficult tasks. In the final reviewed study, Posid and Cordes (2019) compared 3- to 6-year-old CP-knowers' performance on a numerical matching task (with different difficulty levels) when the numerical information was presented either as unimodal (visual only), cross-modal (comparing audio with visual), or bimodal (simultaneously audio and visual) input. The results showed that even the 3- and 4- year olds performed above chance across all three modal conditions. Although this supports the view that young children are able to compare visual and auditory numerical information, the study also revealed a cross-modal disadvantage when the numerical comparisons were easy.

3.2.4 Structured approaches to early learning of numbers: A focus on subitizing

According to Baroody (1987), the essential part of (mathematical) knowledge is structure, which is elements of information systemically connected and organised to form a meaningful whole. A structured-based approach (e.g., dice or patterned arrays) to facilitate exact numbering of small sets is modelled in different theoretical strands, including phenomenology, ecological theories, embodied cognition and neuroscience. For example, a dominant line within neurocognitive-based research on number-space mappings assumes that number magnitudes are represented along a left-to-right-oriented MNL (for reviews, see Newcombe et al., 2015 and; Pixner et al., 2017; see subsection 3.1.1). In contrast, variation theory is a phenomenological inspired learning theory that assumes that patterns of variance and invariance (i.e., differentiation) are fundamental to generalisation and learning (Marton, 2015). A design principle of variation theory is thus to enable the learner to discern multiple patterns of variation of a targeting concept rather than accumulating similar properties. This means to help the child to experience cardinality as a property across spatial configurations, where simultaneous representations of structured finger gestures are identified as a particularly powerful way to discern the cardinal meaning of numbers (Björklund, Ekdahl, et al., 2021). EC explains conceptual formation as a process of abstracting ontological, orientational and structural similarities across embodied experiences (Anderson, 2003; Lakoff & Johnson, 1980). An epistemological implication of EC is thus to facilitate multiple structured embodied experiences that (ontologically) concur with the idea cardinality of numbers (see elaborations in sections 3.1.2 and 4.2). Gibson's (1977, 1979) ecological approach to visual perception emphasises the appropriation of affordances provided by the environment (which basically is a dynamic and moving structure; Braund, 2008). For example, a spatially structured set provides multiple opportunities for direct perception, exploration, manipulation and interaction. However, since only a small portion of the inherent affordances are consistent with a particular learning object (e.g., the idea of cardinality), Gibson (1977, 1979) stresses the importance of a structural analysis of the design (environment) as a basis for guidance towards appropriation of the targeting affordances. General support to the ecological perspective to visual perception of numbers comes from research suggesting a close connection between children's numerical development and visuospatial abilities (e.g., Gunderson et al., 2012; LeFevre et al., 2013; Patro et al., 2014), and studies highlighting the impact of structure and pattern in early understanding of mathematics (e.g., Lüken, 2012; Mulligan & Mitchelmore, 2013).

Several studies have examined the role of pattern recognition and structure in the area of numbers (e.g., McGuire et al., 2012; Mulligan & Vergnaud, 2006; van Nes & van Eerde, 2010; Wolters et al., 1987). The Jansen et al. (2014) study of 4-to 5-year-olds' abilities in exact enumeration of small numbers across three configurations of elements (random, line or dice) shows that the dice presentation in the counting range made pattern recognition significantly easier compared with the two other conditions. However, the configuration manipulation did not affect performance in the subitising range. Schöner and Benz (2017) found that an intervention programme based on a structural approach to numbers helped 5- and 6-year-olds to replace the use of a counting-all-strategy with pattern recognition to determine the cardinality of sets.

The brief overview suggests that there is a growing body of research that contributes to our understanding of structure-based representations in early learning of numbers. Next, I will focus on the line of structured-based inquires connected to the notion of subitising, which per definition is cardinal. The concept of subitising is rooted in ecological theories of (direct) visual perception (see above), but has recently been developed to include multimodal sensory information (cf. the EC perspective).

Subitising refers to the immediate insight of the cardinal value of a small set of objects that might include a verbal response (Sarama & Clements, 2009). The term subitise is derived from the Latin adjective *subitus* (meaning *sudden*) and the Latin verb *subitare* (meaning to arrive suddenly; Kaufman et al., 1949). In contrast to the Latin meaning proposition that the sensory input can be perceived in different modalities, most research has focused on visual perception (Katzin et al., 2019). Data shows that the visual-based subitising limit is age-related, and evidence suggests an upper limit of 3 or 4 for young children and 4 for adults (Anobile et al., 2019), and that the subitising range extends from 3 to 4 around 3 ½ year of age (Starkey & Cooper, 1995) with a higher range for canonical patterns (Katzin et al., 2019).

Although there is some evidence suggesting that early subitising ability is independent of later development of numerical abilities (e.g., Anobile et al., 2019), a wide range of evidence suggests that children able to enumerate small sets without counting perform better in math. For example, in their large scale study, Yun et al. (2011) found that 5- to 8-year-olds' subitising range correlated strongly with their mathematical abilities. This is consistent with studies showing that young children's subitising range correlates with their counting skills (Kroesbergen et al., 2009) and number system knowledge and calculation skills (Penner-Wilger et al., 2007). Desoete and Grégoire (2006) found that low subitising abilities in kindergarten correlated with low math skills in first grade. In their investigation of 7 to 17-year-olds' accuracy and speed of subitising and visual counting, Fischer et al. (2008) found supporting evidence that deficits in subitising predicts difficulties in acquiring basic arithmetic skills. In another related study, Jung et al. (2013) found that pre-schoolers who had received classroom training on spatial features of numerical relationships (i.e., subitising, parts-whole, and more-and-less-relations) scored significantly higher on numbering abilities and conceptual understanding than the control group.

Tucker and Johnson (2018) used a multi-touch technology to explore pre-schoolers' development of embodied subitising. Informed by the notion of conceptually congruent gestures, the investigators operationalised embodied subitising as *all-at-once gestures* in the form of using multiple fingers to mark the entire set simultaneously without explicit use of counting, thereby distinguishing between sequential and simultaneous finger representations of sets. The results of the training with iPads revealed evidence of a close interrelationship between subitising, estimation and (de)composition, indicating that conceptually congruent gestures, and in particular all-at-once responses, are relevant in the early cultivation of number sense.

A study by Riggs et al. (2006) provides evidence suggesting that subitising also occurs in tactile perception. The findings suggested a subitising range of 3 when the students used pressure on the fingertips on both hands as discrete tactile stimuli. There is also limited evidence on auditory-based subitising. To avoid the use of counting strategies, experimental trails present the auditory stimuli in the form of distinct tones in a fast sequence. Within this line of research, the subitising range varies from 2, 3, or 6 for trained musicians, which shows that the findings are unclear (Katzin et al., 2019). In a related study, Anobile et al. (2019) tested if subitising generalises over modalities and stimulus formats for children and adults. The stimuli were presented in the form of simultaneous exposure to visual stimuli (dots), sequential exposure to visual stimuli (flashes) and auditory pitch. The results showed a subitising limit of one item higher for adults than for children across all the stimuli format conditions, and that that the subitising limit for spatial arrays did not correlate for neither of the temporal formats, and that subitising of sequences of sounds and flashes correlated strongly.

3.2.5 Summary

In this section, I review the literature from four interrelated perspectives associated with children's abilities to map exact numbered sets across verbal, visuo-spatial and bodily modalities in an environment requiring (direct) visual perception and physical appropriation of unstructured and structured arrays of dots (cf. focal study 1). First, in subsection 3.2.1, I survey the literature highlighting the verbal modality in children's learning about cardinality, while subsection 3.2.2 provides an outline of the measures used in this dissertation study to assess children's capacities to produce small sets. Subsection 3.2.3 then reviews research on children's abilities to map ideas of cardinality across modalities (cf. the research question in focal study 1), while subsection 3.2.4 reviews the literature on structure-based approaches to early learning of numbers (cf. the arrays used in ETP 1 and focal study 1), with a focus on inquiries related to subitising. To sum up: Combined with the survey in *Article 1* and the review on body-based approaches to number-space associations in subsection 3.1.2, this section provides a comprehensive overview of the field of knowledge associated with children's abilities in grounding the idea of cardinality in multimodal interaction (cf. focal study 1).

3.3 Children's abilities in reasoning about part-whole relations of numbers

This section reviews literature related to children's reasoning about additive compositions, which is the targeting domain in focal study 2 of this dissertation (presented in *Article 2*).

From the 1970s, cognitive researchers began to question Piaget's views of young children's competence in logical and quantitative reasoning as foundational for further mathematical development (Piaget et al., 1952), and researchers focusing on Piagetian skills were replaced by other aspects of mathematical proficiency. More recently, a growing body of scholars view quantitative reasoning abilities along with counting and arithmetic skills, subitised-based enumeration and numerical magnitude understanding as core building blocks of early mathematical proficiencies (e.g., Nunes et al., 2012; Torbeyns et al., 2015). This renewed interest is rooted in the idea that the ability to represent, decompose and compose quantities in multiple and flexible ways, to compare numbers and see numerical relations, is fundamental for later development of sophisticated arithmetic strategies (Björklund, Marton, et al., 2021; Jung, 2011) and for fluency in mathematical problem-solving (Fosnot & Dolk, 2001).

According to Piaget (1952), the aptitude to reason in parts- and wholes, with the exception of *intuitive numbers* (*1-5*), does not emerge before age 6. Recent studies, however, show that pre-schoolers develop more sophisticated mathematical concepts, strategies and skills in abstract reasoning than previously posited (e.g., Clarke et al., 2006; Mulligan & Mitchelmore, 2009). This line of research on early reasoning abilities is influenced by models of numerical cognition that posit a close relationship between the spatial and the numerical domains, and see the existence of spatially organised representations or numerical magnitudes as the core of number meaning (Hyde & Spelke, 2011). Spatial numerical reasoning concerns largely the performance of logical inferences about entities located in space, and these units form a spatial structure (Varzi, 2007). This suggests that the cultivation of number sense and numerosity in form of subitised based visual perception of spatial structures and patterns is fundamental for perceiving parts-whole structures of numbers and for early algebraic reasoning (Mulligan & Mitchelmore, 2013). General support for structure-based approaches in educational settings comes from studies showing that young children are more accurate in solving equation problems in pictures and concrete objects than in numerical contexts (e.g., Gilmore & Bryant, 2006; Sherman & Bisanz, 2009). More specific support comes from the study by Cheng and Mix (2014), which shows that 6- to 8-year-olds who had received spatial training improved significantly better on missing addend problems (e.g., 3 + _ = 7) than children under the control condition. In a related study, Broaders et al. (2007) found that children in elementary school who were instructed to gesture while solving equivalence problems (e.g., 3+4=2+3+_) performed better on equivalence problems in the post-testing compared to children under the non-gesture condition. Moreover, the study by Hunting (2003) provides supporting evidence that 3- and 4-year-olds are able to visualise the missing addend in parts-whole reasoning involving small numbers of items.

FASETT[5] is a Swedish design project on early learning in mathematics that is based on a structural approach to numbers. FASETT builds on principles from variation theory (see subsection 3.2.4), which is a phenomenological inspired theoretical framework of learning (cf. Kullberg et al., 2017). The following studies connected to this project have relevance to my dissertation study. First, Kullberg et al. (2020) 8-month intervention engaging 5-year-olds' in the use of fingers to structure parts-whole relations of numbers in the number range 1-10 reported a significantly higher learning outcome for participators compared to the control group in terms of abilities to recognise and use parts-whole relations in new arithmetic tasks. A follow-up test 1 year later showed that the effect was consistent (Kullberg et al., 2020). However, the delayed post assessment also showed that the 6- and 7-year-olds were unable to apply their knowledge of parts-whole relations to a larger number range, but rather relied on the "double counting" strategy (cf. Fuson, 1988) when bridging through 10 (Björklund, 2021). Finally, Björklund et al.'s (2018) analysis of 4-to-5-year-olds' use of finger strategies in solving simple subtraction tasks (e.g., "If you have 10 candies and eat six of them, how many are left?") identified critical ways of using fingers, from powerful to less effective ways. The less powerful involved procedure-oriented use of the fingers to keep track of counted objects, while strategy effectivity was reflected in more conceptual-oriented approaches in terms of using the fingers to present parts-whole structures of numbers.

Despite a renewed interest in children's quantitative reasoning abilities (e.g., Björklund, Marton, et al., 2021), this central mathematical competence is far less studied than other main areas of children's mathematical development. The limited data available shows that

[5] The ability to discern the first ten numbers as a necessary foundation for arithmetic skills. In Swedish: *Förmågan Att Sinnligt Erfara de Tio första Talen (FASETT) som nödvändig grund för aritmetiska färdigheter*.

proficiency in quantitative reasoning is not easily learned. Therefore, there is a need for more research on young children’s reasoning abilities, and especially studies that examine how reasoning interplays with counting and magnitude understanding (Torbeyns et al., 2015) and how cross-modal representations of numerical relations and structures can support coherence in reasoning about real-life and interdisciplinary problems (cf. Mulligan, 2010).

3.4 Children’s abilities in counting-based addition

This section reviews literature related to the counting-on strategy, which is the mathematical targeting area of focal study 3 of this dissertation study (presented in *Article 3*).

Research shows that children use a wide range of strategies in addition (for reviews, see Fuson, 1992; Verschaffel et al., 2007). Shrager and Siegler's (1998) model on strategy choices of young children suggests at least eight different ways of solving addition problems. The transition from counting to mental-based representations is important for developing fluency in arithmetic (Gersten et al., 2005), and mastery of counting based addition seems to be an intermediate proficiency required for later acquisition of retrieval and decomposition strategies (Clements & Sarama, 2013). To solve simple addition word problems (e.g., *“four plus two”*), most young children integrate knowledge of numbers, abilities in counting and an implicit understanding of addition into informal addition strategies (Levine et al., 1992). In the early stages of strategy development, the most commonly used addition strategy involves finger counting, while verbal counting strategies are used less frequently (Siegler & Shrager, 1984). A conceptual shift occurs when children refine informal counting-all-strategies (e.g., to count both addends in *3+2*) into counting-on from the smallest or largest addend, termed as the *max*- and *min strategy*, respectively (Groen & Parkman, 1972). The min strategy involves stating the value of the largest addend, and then count on the number of times equal to the smallest integer (e.g., “four, five, six” in *4+2*). Similarly, the max strategy (which often involves counting on from the first; Butterworth, 1999) involves stating the smaller integer and successively counting on the larger one. Another major leap in understanding occurs when children begin to retrieve basic arithmetic facts from long-term memory in the form of direct retrieval or decomposition. Direct retrieval involves an immediate response to an addition problem, such as saying “six” when asked to solve *4+2*. Decomposition strategies involve partitioning the addition into a retrievable sum; for example, *4+2* can be solved by retrieving the sum of *3+2* and then adding *1* to this partial sum.

A normal developmental pattern means that rigid addition strategies (e.g., counting-all-strategy) involving finger counting are supplemented and sometimes replaced with more efficient and flexible use of mental-based retrieval and decomposition strategies (Ostad, 1997; Siegler, 1998). However, contrary to this widespread consensus during 30 years of research, Thevenot et al. (2016) found that 10-year-olds' did not retrieve number facts when they solved single-digit addition problems, but rather used fast counting strategies. Based on this, the authors argued that the key change in development of mental addition solving is a shift from slow to more efficient, automated counting procedures (Thevenot et al., 2016).

In an experimental study that taught 5- to 6- year-olds to use a decomposition strategy to solve addition problems, Cheng (2012) found that the children's ability to adopt efficient non-counting strategies was related to knowledge of parts-whole relationships of the numbers *1-10*. The author concluded that appropriate instructional intervention might support children's potential for early acquisition of effective addition strategies. In a related study, Bjorklund and Rosenblum (2001) examined developmental and contextual effects regarding children's multiple and variable use of simple addition strategies while playing a board game using one or two dice to calculate moves. The results showed flexibility in the number and types of strategies used across ages; that children in kindergarten more frequently used the counting-all-strategy than pre-schoolers and first-graders, who tended to vary their solutions with more sophisticated strategies (e.g., the min strategy and fact retrieval). However, the children were unable to transform their knowledge to solve similar oral math problems. Chan et al. (2014) found that young children's counting strategies reflect how much they understand the place-value structure of numbers, suggesting that early intervention of counting-on strategies might have an impact on later arithmetic abilities. In a case study reporting on 6-year-olds' ways of representing understanding of addition, Bakar (2017) found that the participants across representational modes (i.e., concrete material, drawings, gestures) treated the two groups to be added as distinct sets and used the counting-all-strategy to determine the cardinality. The intervention study by Ellemor-Collins and Wright (2009) for low-attaining 3rd- and 4th- graders in addition and subtraction provides case-based evidence that a structured approach to numbers in the range 1 to 20 can result in a significant leap in arithmetical abilities which does not involve counting by ones. This results concurs with the design study by Salmah and Putri (2015), which indicates that ten-structured block activities can support first graders' development of strategies in addition.

In the research literature focusing on finger-based representations to learn basic numerical and arithmetic principles, two opposing views can be identified (Moeller et al., 2011). The first line of evidence has been reported from educational research, advocating a move from finger counting to concrete structured representations, making the foundation for the development of mental-based arithmetic (Moeller et al., 2011). For example, Geary et al. (2004) found that cumbersome finger-counting strategies were related to poor conceptual knowledge of numbers. This line of research is backed up by evidence showing that rigid finger-based strategies are related to learning disabilities in arithmetic (e.g., Ostad, 1998; Ostad & Sorensen, 2007).

Research in neuroscience has provided evidence for a conflicting view, suggesting that finger-counting strategies play a crucial role in the development of fluency in arithmetic (Berteletti & Booth, 2016; Butterworth et al., 2011). The basic argument for this stance is that the use of finger representations provides multi-sensory input and thereby facilitates encoding of cardinal and ordinal aspects of numbers (Moeller et al., 2011). Support for this claim comes from neurocognitive data showing that finger gnosis (i.e., the ability to differentiate one's own fingers without using the visual sense) is associated with children's arithmetic skills (e.g., Noël, 2005; Penner-Wilger et al., 2007). Also, several scholars highlight the benefit of using finger patterns to promote children's structural awareness and pattern recognition of parts-whole and cardinal-ordinal relations of numbers, rather than using fingers to count single units (e.g., Baroody, 2016; Neuman, 2013). Empirical support for this view comes from the study by Björklund, Ekdahl, et al. (2021), which shows that 4- to 7-year-olds' use of finger gestures for simultaneous representations of small numbers might support perception of sets as structured wholes rather than strings of single units, thereby facilitating arithmetic thinking. Hence, this suggests that there are quality differences in the use of finger gestures to promote mathematical thinking, where the powerful ways are characterised by simultaneous experiences of cardinal and ordinal properties of numbers as a basis for fluency in arithmetic.

3.5 Summary of the literature review

The following two issues related to my dissertation study emerged from the literature review in this chapter. First, the review of number-space mappings from the cognitive and interactionist stance of EC shows that there is a large body of research that provides general support for the moving and active body in early learning of mathematics (section 3.1). However, due to a bias towards modelling the MNL in educational interventions, I question the transfer effect to practice in ECEC institutions. Second, the review of the literature on children's understanding of cardinality and abilities in parts-whole reasoning and addition (sections 3.2 - 3.4) provides an overview of the current field of knowledge of how different modalities and representations of numbers and relations might influence children's mathematical abilities (cf. the studies assumptions and research questions asked in sections 2.5 and 2.6). Notably, across these mathematical areas, there is an emergent body of research that views structural relationships of numbers as foundational for the development of mathematical abilities. In conclusion, the review provides many reasons to conduct intervention studies with young children that include structure-based embodied experiences of cardinality, ordinality, and parts-whole and additive relations of numbers (cf. sections 4.3 and 5.2).

4. Theoretical framework

This chapter starts with a brief historical overview that foregrounds EC followed by an outline of main features of the EC framework. It concludes with a presentation of the design principles that form the basis of the two embodied training programmes.

4.1 Introduction to embodied cognition

Classical philosophy and traditional cognitive science (i.e., Cartesian cognitive science, cf. Rowlands, 2010) have tended to marginalise the role that the body-beyond the brain, action and environment play in thinking, reasoning and cognitive development, assuming that sensory-motor processes only function as peripheral input and output devices (Wilson, 2002). Early critiques of the mind-body separation include Kant who argued that mind is a manifestation of the body (Kant & Guyer, 1998). Later, phenomenology contested the idea of a disembodied rationality that separates thinking from sensing and acting, arguing that there is no such thing in how we perceive, experience and act in the world that supports such a dualism (Husserl, 1970). Notably, Merleau-Ponty (2005) assigned the body a mediating role between internal and external experiences when he explored the relationship between embodied action and meaning, claiming that our body "keeps the visible spectacle constantly alive, it breathes life into it and sustains it inwardly, and with it forms a system." (Merleau-Ponty, 2005, p. 235). This shift, which Clark (1998) refers to as putting the brain, body, and world together again, shows that embodiment is a theme addressed in philosophy during the last century.

Influenced by the ideas of phenomenology, several cognitive scientists tried to deal with the troublesome disembodied nature of cognition (e.g., Churchland, 1989; Wertsch & Wertsch, 1993). However, logical limitations in dualistic views of mind-body and cognition-action were first and foremost made apparent by the symbol grounding problem, where Harnad (1990) rhetorically asks how the meaning of symbols seen as internal mental representations can be grounded in anything but other abstract symbols whose existence are independent of time and space. Based on this line of thinking, higher-order cognitive representations must be rooted in non-mental, physical and embodied representations (Barsalou, 1999). In addition, traditional cognitivism failed to explain the emergence of abstract representations and how and where these abstract ideas are implemented in the cognitive system (Barsalou, 2008; Gallese & Lakoff, 2005). Consequently, to deal with the symbol grounding problem and to provide adequate explanations of epistemological issues concerning representation, retrieval and concept formation, cognitive models must incorporate how mental ideas relate to embodied experiences and physical properties.

4.2 Main features of embodied cognition

In this section, I will first present the main features of EC and then focus on the principles relevant to the extraction of the design principles (section 4.3) and for the discussion (Chapter 7)[6]. For a comprehensive overview of the EC framework and its implication for educational science, I recommend "The Routledge Handbook of Embodied Cognition" (Shapiro, 2014), the special issue "Embodied cognition and Education" (introduced by Agostini & Francesconi, 2020), and the article "Embodied cognition and its significance for education" (Shapiro & Stolz, 2019).

The modern version of EC is informed by recent research in cognitive science, ecological psychology, animal science, neuroscience, linguistics and different domains in learning sciences and educational research (Agostini & Francesconi, 2020; Barsalou, 2008; Wilson, 2002). Moreover, different theories of EC vary in terms of the boundary set for mental processes and according to what claim the body-cognition relation addresses (Wilson & Foglia, 2017). Accordingly, no unified proposal exists for EC as some researchers suggest two (Stevens, 2012; Wilson & Golonka, 2013), others three (Shapiro, 2011) or even six (Wilson, 2002) different views of EC. Wilson and Golonka (2013) argue for a two-part view of EC, one stance that focuses on how mental states and processes can be influenced by the body, and another that focuses on the brain-body-environment relationship in cognition. This is consistent with Stevens' (2012) notions *embodiment as conceptualist* and *embodiment as interactionist* as two main strands of EC. This suggests that the diverse families of EC recognise a range of perceptual, cognitive, and motor processes that are grounded in the capacity of the physical body (Fugate et al., 2018). However, while recognising rich variations, most EC frameworks agree on two main features: 1) Cognition is inextricably linked to sensory-motor processes and bodily interactions with the environment, and 2) such embodied interactions are mentally represented in a non-abstracted manner (e.g., Barsalou, 2008; Borghi & Pecher, 2011; Shapiro,

[6] The outline of EC builds on and must be seen in connection with the presentation in section 2.2 ("The turn towards ideas of embodiment in educational research") and section 2.5 ("The studies assumptions: Towards the aims of the dissertation study").

2011). Additional principles of the EC framework that are only partially true are the claims that cognition is situated, for action and subject to time pressure, and that human reduce the cognitive workload by off-loading cognitive processes onto bodily and environmental resources (Wilson, 2002). Accordingly, these claims are best weighted in terms of their applicability in practice-oriented research.

From an EC perspective, the formation of abstract concepts rests on the perceptual and sensory-motor system (e.g., auditory, visual, kinaesthetic, and haptic-tactile) that captures embodied interaction and serves the needs of a body interacting with a real situation. Accordingly, on-line aspects of cognition are affected by real-time demands, spatial constraints and contextual affordances and is therefore fundamental for situated performance and learning (Nathan, 2008). Later, independent of the tangible stimulus, thinking about an action through the process of re-enactment (i.e., mental simulation and visualisation) will induce the same multimodal sensation that occurred during the actual embodied experience (Barsalou, 2008). Such off-line cognitive processes range from pure mental simulations involving sensorimotor representations to the use of external resources to support mental representations and manipulations of things that are distant in time or space. Accordingly, a core feature of cognition involves the ability to re-enact and model situated sensory-motor experiences in off-line mode (Nathan, 2008). Such mental simulations often occur unconsciously in an automatic manner, and can be partial, incomplete, and convey misunderstandings (Barsalou, 2003; Wilson, 2002).

EC posits that conceptual knowledge is rooted in many simulations, specific to particular instances or embodied experiences with the stimulus. For example, sets can be understood as *containers*, numbers as *objects in a container* and arithmetic can be understood as *making groups/object collection* and *moving along a path* (Lakoff & Núñez, 2000). Consequently, no specific or individual simulation might give a complete representation of a targeting concept (Radford, 2013). Based on this line of reasoning, mathematical thinking is grounded in multiple overlapping and mutually constitutive simulations (Barsalou, 2003, 2008). Also, EC entails a perception-action cycle involving the succession of motor adaptation to changes in external (e.g., attachable objects, body posture) and internal (e.g., mental connections) *environment* (Fuster, 2009). This continuous and bidirectional influence between conceptual and external (beyond-the brain) activation generates feedback for regulating further actions. The embodied approach can therefore be fruitful in connecting mathematical thinking in physical interaction with the world (Barsalou, 2008; Wilson, 2002).

To conclude, EC constitutes a theoretical framework for educational science and practice (Agostini & Francesconi, 2020), and its multidisciplinary nature can provide some thought-provoking recommendations that can enhance educational practise in a way that supports quality and efficiency in children's learning (Shapiro & Stolz, 2019). Based on this, the next section presents the DPs of the ETPs included in this dissertation study.

4.3 Design principles of the embodied training programmes

This section outlines key concepts and issues concerning design principles in educational settings, and it provides a brief survey of the literature as a basis for the following presentation of the principles that guided the design, implementation, evaluation and modification of the activities included in the embodied training programmes of my research project.

4.3.1 Key concepts and literature review on embodied design principles

A pedagogical model is informed by theory of learning and instruction, and it accounts for the interdependent relation between learning, facilitation/guidance, subject matter and context. A basic assumption for educational design aimed at optimising thinking and learning is to look at fundamental research in cognitive science that provides an accurate description of the nature and function of cognition (Ionescu & Vasc, 2014). From an embodied perspective, this means to apply theory of EC in the design of learning environments that foster grounded thinking and learning that encompass tacit and cultural ways of perceiving and acting (Abrahamson, 2013; Abrahamson & Lindgren, 2014). In DBR, design principles play a key role in establishing the relationship between an implementation of a design and the theoretical conjectures informing the pedagogical model (Sandoval, 2014).

Through my review of literature using different combinations of the keywords *mathematics*, *outdoors*, *embodied design*, *design-based research* and *design principles*, two relevant observations connected to my dissertation study emerged.

First, most of the surveyed literature on embodied design principles was situated in digital contexts in indoor settings (e.g., Abrahamson & Bakker, 2016; Dackermann et al., 2017; Johnson-Glenberg, 2018). For example, DeSutter and Stieff (2017) suggest three design principles developed to incorporate movement for the learning of mathematics in digital environments. These are:

> "Design principle #1: Embodied learning environments should include scaffolds that explicitly map spatial entities and their relationships to the hands or the body… Design principle #2: Embodied learning environments should leverage motoric actions to simulate high-fidelity spatial operations that would otherwise be imagined… Design principle #3: Embodied learning environments should link innovative tools, such as visualizations or other simulations, to embodied actions through interface elements and input devices" (DeSutter & Stieff, 2017, pp. 12-13)

Moreover, Johnson-Glenberg's (2018) taxonomy for embodied designs using VR technology includes the following guiding principles; (1) Sensory-motor engagement; (2) Congruency of the gesture; and (3) Immersion/Presence. In Skulmowski and Rey's (2018) review of taxonomies in embodied research in education, the authors argue that Johnson-Glenberg's (2018) three suggested factors are not optimal descriptive dimensions for embodied learning studies (e.g., that the category perception of immersion lacks an empirical foundation). Skulmowski and Rey's (2018) alternative and more general proposal consists of a 2x2 grid involving the main dimensions (1) bodily engagement (i.e., low or high degree of bodily activity) and (2) task integration (i.e., whether the bodily activity coheres with a learning objective in a meaningful way or not). The first of these principles is consistent with what DeSutter and Stieff (2017) refer to as learning environments that allow bodily mapping of spatial entities and relations, while the second principle is comparable with what Johnson-Glenberg (2018) refers to as congruency of the gesture principle. Based on Skulmowski and Rey's (2018) taxonomy, I position the DBR part of my dissertation study in the quadrant involving high degree of bodily engagement and high degree of task integration.

Second, the survey of the literature on design principles in outdoors settings showed that many of the proposed design principles were of a general pedagogical character and consequently not directly relevant for my research focus on epistemic processes in body-based learning environments. For example, in the review study by Mansfield et al. (2020), several of the ten identified principles of effective youth development in outdoor environments, with the exception of utilising adults in guidance and mentoring and to develop skills through authentic experiences, were considered to be only partly or indirectly related to my research project. These principles include the benefits of a positive social context, and allowing the children to work for an extended period and within a continuum of activities. Other salient principles, however considered outside the scope of my research, include opportunities to develop autonomy, the use of unfamiliar environments and opportunities for reflection. This points to limitations that any selection of design principles will put focus on certain epistemic commitments and therefore indirectly hide other salient features affecting young children's learning. However, having established that some of the reviewed principles can be adapted to non-digital outdoor contexts using whole-body movement, the next subsection outlines the design principles used in my dissertation study.

4.3.2 Design principles guiding the (re-)design of my embodied training programmes

I extracted the DPs used in my research project from four interrelated perspectives and sources. These were: (1) The studies assumptions and aims (sections 2.5 and 2.6); (2) The EC framework (section 4.2); (3) Recent findings in educational research (cf. the literature review in sections 3.1-3.4); and (4) Embodied design studies in education (subsection 4.3.1). Thus, in order to make a coherent foundation for designing activities that takes into account learning, design and instruction, the development of the ETPs and the testing procedures (cf. the focal studies) included in this dissertation study were informed by the following DPs (DP 1 – DP 4):

1. *Congruence in number-space mappings through embodied interaction.* DP 1 builds on educational research that highlights the role of structure-based approaches to young children's understanding of mathematics (e.g., Venkat et al., 2019) and especially the line of research that focuses on spatial thinking of numbers, parts-whole relations and arithmetic through embodied interaction (cf. the literature review in Chapter 3). To support (high degree of) congruence with the mathematical targeting domains, the designs should foster highest possible overlap in feature codes by presenting the stimuli in a spatially organised perceptual format that allows for full-body spatial interaction and manipulation with tools (DeSutter & Stieff, 2017; Fischer et al., 2011). Hence, DP 1 highlights that the bodily action should correctly simulate (structure-based) mathematical thinking (cf. the principle of task integration of Skulmowski & Rey, 2018).
2. *Meaningful grounding of mathematical thinking in the three-dimensional space.* DP 2 reflects the governing principle of embodied design in terms of situating mathematical thinking in authentic ways of perceiving and acting (Abrahamson & Lindgren, 2014). This means that the design should foster meaningful experiences by rooting the mathematical thinking in cultural, motoric and ecologically rich behaviour (Johnson-Glenberg, 2018),

thus cultivating the phenomenological dimension of meaning through the idea that cognition is for action (Wilson, 2002). Accordingly, DP 2 reflects the assumption of EC that many everyday experiences (e.g., body postures, gait, object construction/collection) and play behaviour can provide both subjective and objective layers of meaning to mathematical thinking (Lakoff & Núñez, 2000; Radford, 2010, 2013). In that way, DP 2 is partly overlapping and complementary to DP 1's emphasis on the objective dimension of meaning in mathematical thinking (cf. the congruence principle). Also, DP 2 includes the assumption that (meaningful) number-space mappings (including the OTS and ANS) occur in three dimensions (i.e., not linear modelling according to the MNL; Winter et al., 2015).

3. *Multimodal experiences*. DP 3 builds on the fundamental assumption of EC that sensory-motor experiences in the form of multimodal simulation matching a targeting domain might enrich encoding and thereby facilitate the re-enactment of these ideas in the form of mental simulations and visualisations (Barsalou, 2008; Carbonneau et al., 2013; Wilson, 2002; cf. section 4.2). Furthermore, it was conjectured that the interplay between the bodily and verbal modality played a key role in making non-verbal number-space associations explicit as the "grounding aspect of the body acts as a *scaffold* for articulating thoughts that otherwise would be difficult to communicate" (Foglia & Wilson, 2013, p. 4).
4. *Guidance and learning through imitation*. Previously, I have argued that children need guidance to pay attention, discern and appropriate exact numerosities and mathematical relations in their outdoor environment (cf. section 2.3). Support for DP 4 comes from the review study by Mansfield et al. (2020), which showed that a key characteristic of effective youth development in outdoor environments is the use of adults to guide and mentor. Also, mirror neurons that activate in the same way when we carry out an action with our body or observe the same action performed by others suggests that imitation of motor behaviour and bodily states is fundamental for learning (Rizzolatti & Craighero, 2004). Consequently, DP 4 emphasises that the designs should foster learning through guidance, observation, mirroring and imitation.

DP 1 reflects my objective of designing ETPs that foster spatial structured bodily experiences that concur with thinking about cardinality and counting-based addition as subject matters (in ETP 1 and 2, respectively; cf. the aims stated in section 2.6). As complementary epistemological principles in my suggested pedagogical model, DP 2 and 3 reflect that engagement within the design should involve meaningful use of motor, cognitive and cultural resources in the form of multimodal experiences matching the content area and thus enrich learning and retrieval.

Finally, DP 4 emphasises facilitation, guidance and imitation as supporting elements for the processes mentioned above. This suggests that the DPs are assumed to be mutually connected, reinforcing and partly overlapping, and that they together make a coherent foundation for (re-)designing activities promoting children's grounding of mathematical thinking. However, the principles are limited in the sense that they do not sufficiently take into account the value of including reflection or social, motivational and other general pedagogical principles in the design of the activities (cf. Mansfield et al., 2020).

5. Research design and methods

In this chapter, I will reflect upon methodological issues of my dissertation. First, the choice of a qualitative research strategy is elaborated, including the design-based research and case study approach. It then outlines methodological issues related to the design of the embodied training programmes, and it presents information of participants, samples and data collection. Thereafter, the approaches to achieve generalisation, techniques for analysing data, the process of data reduction and coding are elaborated in detail. The final part presents reflections on the quality of the study including issues about trustworthiness, ecological validity and ethics.

5.1 Methodological choices and research design

The research questions were the starting point for selecting the research methodology of the dissertation, as the research methods were to provide data to answer the questions asked (Maxwell, 2008). Basic aspects of qualitative studies include the in-depth focus of small groups of participants and the use of soft, flexible and subjective data in order to investigate the participants' behaviour, meanings and reasoning around specific subjects (Silverman, 2010). In this regard, a qualitative approach was selected because it made it possible to study in detail characteristics of young children's grounding of mathematical thinking in sensory-motor experiences (cf. the framing question of the dissertation). Furthermore, there is consensus among prominent researchers in the methodological field that the synthesis of several methods can compensate for individual limitations and benefit from their respective strengths (Guba, 1981; Yin, 2009). Based on this, my dissertation study adapted principles from *Design Based Research* (DBR) and case study methodology to answer the research questions.

The most important argument for DBR stems from an ambition to strengthen the link between practice and research (Van den Akker et al., 2006). In this regard, the dissertation study aim to develop ETPs that can be integrated into the daily practice of ECEC institutions (cf. the sub-goal of the dissertation). Also, in line with the main objective of DBR, the dissertation aims to refine principles of EC by producing and studying empirical data on learning processes, use of tools, guidance and facilitation (DiSessa & Cobb, 2004). The learning part is addressed in the three focal studies and in the discussion chapter of this dissertation, while complementary reflections concerning the role played by the ETPs in producing epistemic achievements are a central aspect of the current methodological chapter.

In this dissertation study, I use the EC framework to develop the DPs (section 4.3), the research questions (section 2.6) and analytical tools (subsection 5.4.1), and for interpretations of the empirical data from the focal studies to achieve analytical generalisation (see sections 5.3 and 5.4). The focus on analytical generalisation is consistent with Yin (2009), who argues that the goal of a case study is not to achieve statistical generalisation, but rather to confirm, expand and generalise theories. The focal studies reported in the three articles (see Chapter 6) apply a multi-case study approach with the aim of gaining an in-depth understanding of early grounding of mathematical thinking in sensory-motor experiences. This is consistent with the basic idea of multi-case studies, which is to use any appropriate method to examine one or several cases in detail in order to gain a holistic understanding of the case(s) (Silverman, 2013; Yin, 2009). In this way, the multi-case approach allows me to study similarities and diversities across children's bodily grounding of mathematical thinking, which in turn enhances contrasting results and reinforces similar results (Yin, 2009). Compared to single-case studies, the diverse empirical material from the multi-case studies and the use of the logic of replication and comparison allow me to conduct a broader exploration of the investigated phenomena, which in turn is fundamental to the robustness of theory development (Eisenhardt & Graebner, 2007; Firestone, 1993). Consequently, evidence from multi-case studies is considered strong and reliable (Baxter & Jack, 2008).

Figure 1 below provides an outline of the research design of my dissertation study.

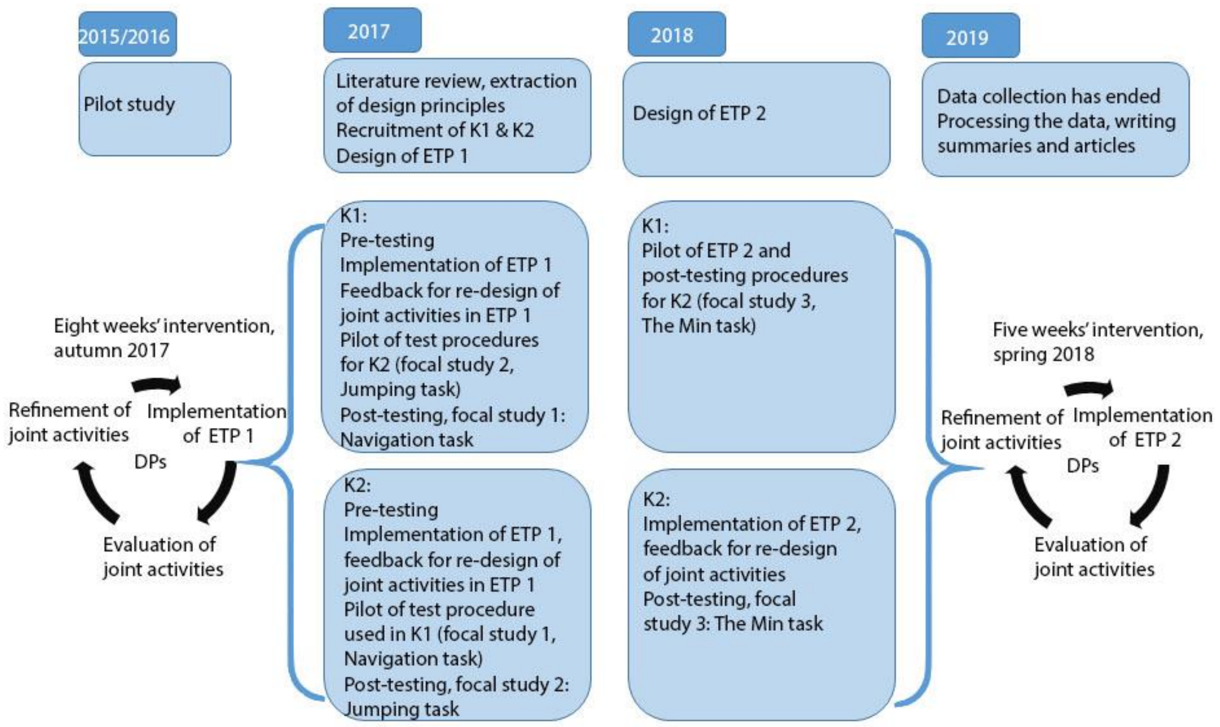

Figure 1: Overview of research design

In 2017, preparations for the research project were made by recruiting participating kindergarteners and reviewing the literature on EC and early childhood learning in mathematics for the extraction of the DPs presented in section 4.3. The development and refinement of ETP 1 and 2 was based on these DPs, while ETP 1 also was informed by a pilot study (Appendix 1). Both ETP 1 and 2 involve a set of joint activities and a fixed task. Through the interventions and based on video analyses of sessions, logged reflections and informal dialogue with KTs, small modifications of the joint tasks were made in a cyclical analytical process with testing, evaluation and refinement, while the fixed task was repeated at all sessions. The post-test procedure in focal study 1 (Kindergarten 1) was piloted in Kindergarten 2, and vice versa for focal study 2, while Kindergarten 1 served a pilot function throughout ETP 2 and for the development of the post-test procedure for focal study 3 (Kindergarten 2). The empirical material collected for the focal studies consists mainly of video of individual post-tests. The research design is elaborated later in this methodological chapter.

5.2 Foundation of interventions and (re-)design

This section focuses on the design part of my dissertation study. First, I will present the activities included in the two ETPs followed by examples of how I applied the DPs in the development of these activities. I will then outline issues related to the cyclical relation between testing and re-design, and I will conclude the section with a presentation of the practical implementation of the programmes, which includes an outline of the KTs' and the researchers' roles and responsibilities.

5.2.1 Activities included in the two embodied training programmes


Figure 2: Arrays used in the joint activities in ETP 1


Figure 3: Arrays used in the fixed task in ETP 1


Figure 4: The construction activity

Each session of the ETPs included a joint part (approximately 20 minutes; cf. Figure 2) and a construction activity (approximately 40 minutes) where the children should use their bodies to simulate a mathematical targeting domain (see Figure 3) to get a piece to play with (see Figure 4). In line with this structure, Table 1 presents a list of the activities in the two ETPs referred to in this dissertation, while an elaborated outline of the activities follows below.

Table 1: Overview of activities used in ETP 1 and ETP 2

	ETP 1	ETP 2
Joint activities	*Animal farm* *Seek and embody numbers* *Rhythmic tagging*	*Tag a number* *Counting-on x more* *Hit the home-base* *Rhythmic tagging*
Fixed task	*Imitate animal*	*Min task*


Figure 5: Long jump for enacting *Cock-a-doodle-doo-one*


Figure 6: Making twists in embodying *Monkey-three*


Figure 7: *Cock-a-doodle-doo-one* on a stump

Activities used in *Embodied Training Programme 1* (ETP 1)

Animal farm (joint activity)

The children were assigned a square array outside a semicircle (see Figure 2) with the researcher at the centre and KTs in support nearby. The context of the story was a farm with various animals, including a rooster (*Cock-a-doodle-doo-one*), a kangaroo (*Kangaroo-two*), a monkey (*Monkey-three*) and a frog (*Frog-four*)[7]. The researcher started the story:

> It is early in the morning, the sun is rising (pointing at the horizon) and everyone living on the farm is sleeping, except one animal. Who is that? … Yes, the rooster. What does the rooster do? …Yes, the rooster stands on one leg and crows "cook-a-doodle-doo-one" (the researcher balances on one leg while tagging a dot). Therefore, to wake up the animals and people on the farm, we must all be crowing roosters. Ready, steady, go: Cock-a-doodle-doo-one. (Everyone shouts "Cock-a-doodle-doo-one" balancing on one leg that marks a dot). Did all the animals wake up? No (shaking the head). We have to jump longer and higher and we have to crow even louder, much louder. Are you ready? One, two, three... (See Figure 5) [The physical mapping continues].

After several attempts, even the great-grandmother with bad hearing was awakened. Similarly, the children reflected upon the behaviour of kangaroos, monkeys and frogs, and they used limbs (i.e., feet and hands) in simulations of the animal's behaviour in physical couplings of 2-to-4-dotted arrays. In particular, the *Kangaroo-two* mapping involved both feet in the coupling of two dots, the *Monkey-three* gestalt enacted the number three by using the legs and one hand (the other hand scratching the head) in the coupling of a three-dotted array (see Figure 6), while the *Frog-four* mapping embodied a square array with both legs and hands. Based on the animals' desire to have fun, the children were encouraged to be creative, make long jumps, twists and rotations before entering into the arrays, while making exaggerated articulations.

Seek and embody numbers (joint activity)

The children worked in pairs in the playground, looking for detachable and attachable objects on the ground (e.g., pinecones, stones, leaves, sticks, stumps and cobblestones). Based on observed arrays, they were encouraged to use animal metaphors as verbal references to explore creative ways of embodying these spatial structures (see Figure 7).

[7] Translated from the Norwegian words "kykeliky-en", "kenguru-to", "apekatt-tre" and "froske-fire".

Figure 8: Dancing to the *Gummy Bear song*

Figure 9: Free-dance to the *Gummy Bear song* (from ETP 2)

Figure 10: *Eeny, meeny, miny, moe, one, …, four* (from ETP 2)

Rhythmic tagging (joint activity used in both programmes)
The children were guided in the aerobic movement pattern involving “Right leg forward, left leg forward, right leg backward, left leg backward” to sequentially tag dots in the square array (see Appendix 1 for details). The cyclical movement pattern was accompanied by music (e.g., *the Gummy Bear Song*; see Figure 8, which also involved a free dance part; see Figure 9) and rhymes (e.g., *Eeny, meeny, miny, moe*; see Figure 10) streamed from YouTube. The aerobic movement pattern was also used in combination with verbal counting up or down (e.g., “One, two… eight” or “Four, three, two, one”) or repeatedly counting (e.g., “One, two, three, four [short pause in one-legged position], one, two, three, four …”). Variation in the auditory modality could be experienced via simulation of different animals (e.g., elephant with hard and loud tramps, mouse with softer ones).

Imitate animal (Fixed task)
To get an item to play within a construction activity, the children should in increasing order articulate the animal metaphors (i.e., “Cock-a-doodle-doo-one”, “kangaroo-two”, “monkey-three” and “frog-four”) in parallel with the physical coupling to the corresponding array (canonical structure for 3 and 4; see Figure 3). After completion, the children could choose one item from boxes of toy figures, animals and construction components (see Figure 4).

Figure 11: Everyone in tune, counting to ten

Figure 12: The chase starts

Figure 13: Counting-on from tagged number to get released

Activities used in *Embodied Training Programme 2* (ETP 2)

Tag a number (joint activity)

Everyone in tune with verbal counting while moving on dots in the 100-dotted circle (see Figure 11), the chase started at ten (see Figure 12). The children could run within a limited area, and tagged children could re-join the game after using the circle for counting-on to ten from the number they had been tagged with (Figure 13).

Counting-on x more (joint activity)

In turn, the children rolled a dice (e.g., 5) and decided how many they should count on (e.g., two more). Based on this and starting outside the 100-dotted circle, everyone was to use his or her feet for synchronised verbal and physical expression of the value of the dice (e.g., tramping on the ground while saying "Five"). Next, inside the circle, everyone was to, in tune, walk on dots and verbally count-on, while balancing on one leg when they reached two/three/four more. While standing on one leg in the final tagging, the children were encouraged to shake hands with peers and articulate the sum.

Figure 14: A pea bag as external motivation for *counting-on* to ten

Hit the home base (joint activity)

Two children could work together using a dice and a pea bag (or tennis ball) as tools. In turn, the children were to roll the dice, enter into the 100-dotted circle and use their legs to tag dots for counting the outcome of the dice or for counting-on to ten. In the final tagging, they were to balance on one leg while throwing the pea bag (alternatively, rolling the tennis ball) to their peer or hitting the home base (a small circle) if they worked alone.

Figure 15: The *Min task* to get a piece for construction and play

Min task (Fixed task)
To pick an item to play with from containers of matchbox cars, toy figures and construction components, the children had to complete the following steps: 1) Roll two dice (e.g., 3 and 5), compare the values and pick up the one with the smallest addend. 2) Articulate and physically tag the largest addend as a whole (e.g., use the feet tagging the dice and say “five”). 3) Enter the 100-dotted circle, and informed by the value of the hand-held dice, using their feet to tag dots and verbally count on the number of times equal to the smallest addend (e.g., “six, seven, eight”). 4) In the final tagging, they were to keep balancing on one leg while articulating the sum (e.g., “eight”).

Table 1 shows that the two ETPs include a limited number of activities. Instead of working with a large number of activities, the intention was to let the children familiarise themselves with the tasks as they gradually went deeper into the selected mathematical topic. Put together, involvement in the two ETOs was assumed to promote coherence in embodied interaction and mathematical thinking. Below, I will give examples of how I applied the DPs in the development of the activities in the ETPs.

5.2.2 Applying the design principles to the two embodied training programmes

Although the DPs in section 3.4 are presented as distinct dimensions, it is important to note that they are assumed to form a coherent foundation for designing activities that foster bodily grounding of mathematical thinking. Below, I will give examples of how I integrated the four design principles (referred to as DP 1 to DP 4) into the activities.

1. Congruence in number-space mappings through embodied interaction
2. Meaningful grounding of mathematical thinking in the three-dimensional space
3. Multimodal experiences
4. Guidance and learning through imitation

The structure-based approach to facilitate coherence in children's embodied modelling of the targeting mathematical domains (cf. DP 1) was reflected in the design of the arrays. For example, the 1 to 4 dotted arrays in the *Imitate animal* activity in ETP 1 aimed to foster symmetric structured experiences where body parts touching the ground should cohere with the idea of cardinality. Another principle was that the children should experience some sort of subjective meaning in their bodily engagement (cf. DP 2). To address both these dimensions, each joint session involved a game and/or a song targeting specific mathematical content areas (see subsection 5.2.1). For example, the *Rhythmic tagging* activity involving the aerobic movement pattern in tune with music, aimed to enhance joy, shared experiences and awareness of one-to-one-correspondence between kinaesthetic, spatial, aural and verbal modalities, thus adding both personal, social and objective meaning dimensions to key subcomponents of what a multimodal approach to exact numbering and counting-based addition entails (DP 3). Another illustration comes from ETP 2, where the *Hit the home base* activity challenged the children in precision throwing while balancing as a part of their bodily engagement of core building blocks of counting-based addition (DP 1 - DP 3). A third illustration comes from the same programme, where the *Tag a number* activity wrapped multimodal experiences in counting-on from a given number in a game with strong cultural traditions in outdoor play for Norwegian children (DP 1 - 3). In contrast, the *Min task* in ETP 2 was based on more external motivational factors as the children were obliged to work with the consolidation of counting-based addition to achieve components for their construction activity.

DP 3 involves the fostering of multimodal experiences of the targeted mathematical content area. The *Imitate animal* activity in ETP 1 might illustrate this principle as it entails the physical coupling of one to four-dotted canonical patterned arrays while articulating a corresponding metaphor (cf. also DP 1). Another example comes from ETP 1, where the *Seek and embody numbers* activity aims to promote children's curiosity and awareness of quantifiable properties in their (three-dimensional) surroundings, using *animal metaphors*[8] as verbal references for exact bodily numbering of small sets (DP 1 - DP 3).

A shared feature of both programmes was the wrapping of the fixed tasks in a construction activity where the children could collaborate, play, and design whatever they wanted (cf. DP 2). Help to perform the obliged task (i.e., the *Imitate animal* activity or the *Min task*) to get a piece for their play or construction was only provided if needed. Overall, the guidance structure across the intervention sessions was based on working with subskills as a basis for more complex multimodal integration at later stages, which also involved demanding bodily actions (e.g., body rotation before the tagging of arrays; cf. DP 2 and DP 3). Another guiding principle involved bodily and concrete modelling for KTs and researcher to provide direct and meaningful feedback on children's performance and that the children should learn through observing the action of peers (DP 4). For example, in the joint *Animal farm*-activity in ETP 1, the process of mimicking the behaviour of animals living on a farm aimed to promote motivation, realism and meaning to the embodied action (DP 2), while also fostering learning through observation and copying other children's behaviour (DP 4).

To sum up: The ETPs show how the DPs might be the starting point for developing a battery of activities that promote young children's embodied grounding of mathematical thinking. Since the degree to which a DP applies to an activity can vary, it is important to see each ETP as a whole that together are conjectured to promote meaning for the individual's physical situating of mathematical thinking in outdoor contexts. Below, I will outline issues related to the evaluation, modification and re-design of the activities in the ETPs.

5.2.3 Evaluation and re-design of the embodied training programmes

DBR is characterised as a dynamic, evolving and iterative process of testing, evaluation and refinement (Schön, 1983), and this inductive, reflexive and emergent endeavour underlines the explorative dimension of my dissertation. However, the fixed tasks in the ETPs (i.e. the *Imitate*

[8] The dissertation's use of the notion of *animal metaphor* is based on CMT, which holds that metaphors are conceptual phenomena that help us to transfer meaning and carry inferential structures from a source domain to a target domain (see *Article 1*). For example, the *Frog-four* metaphor grounds the idea of cardinality in a particular body posture that shares some abstract similarities with amphibian behaviour (e.g., the cardinal value of limbs touching the ground). Gallagher and Lindgren (2015) refer to this as *enactive metaphors*.

animal-activity and the *Min task* in ETP 1 and 2, respectively) were the same throughout the intervention, while the joint activities followed the basic principle of DBR concerning evaluation and refinement across sessions. In general, any refinement was guided by the DPs (see section 4.3), and the motivation for change could emerge from various sources including feedback from children (e.g., bodily and verbal responses regarding pleasurable games and songs) and critical reflections from the collaborating KTs. However, based on the terms of agreement made with the participating kindergartens, resources to the KTs were only assigned for implementation and not for planning and evaluation. To address this challenge, I used a notebook to log discussions of the activities with the KTs at available periods during sessions. In addition to this unstructured approach, I used video recordings of the activities for assessment post sessions. Finally, the implementation of both training programmes in two kindergartens enabled me to transform experiences across sites as a basis for modification.

A large part of my role as a researcher was to design joint activities that addressed subskills associated with the targeting mathematical domains in the respective programmes and to focus on areas identified as difficult for the children to master in previous sessions. For example, the *Counting-on x more* and *Hit the home base* activities in ETP 2 were designed during intervention to address core building blocks of counting-based addition. Furthermore, the inclusion of artefacts such as pea bags and tennis balls and motoric cooperative behaviour (e.g., shaking hands while balancing on one leg or throwing and catching pea bags) emerged from observations that pointed on a need to include additional layers of meaning to the activities (cf. DP 2). Such small and iterative modifications continued throughout the intervention period, and I will illustrate the refinement-cycle of the *Tag a Number*-activity in ETP 2 in detail. In the first two sessions, a joint activity involved the use of the counting-on strategy to find the total age of two children. Informed by their verbally expressed ages, the children were supposed to step into the 100-dotted circle and in tune count-on from the largest age the number of times equal to the smallest age in order to determine the sum. However, this activity was rejected as it became too abstract and the children lost engagement. Thus, in order to simplify and motivate, I wrapped the counting-on part in the game *Tag*. In the first issue of the *Tag a Number* activity, tagged children had to run to the 100-dotted circle, toss a dice, and then count-on to ten in order to continue the game. However, to enhance flow, the chaser tagged the children with a number 1 to 6, which they used as a basis to count-on to ten.

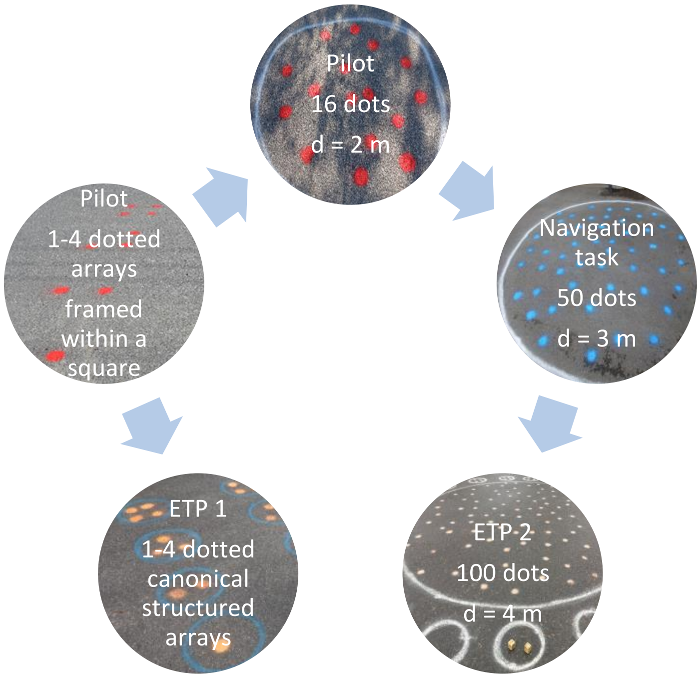

Figure 16: Refinement cycles of array's used in the ETPs and in the post-tasks

Although I did not modify the fixed tasks during the interventions, the designs used in ETP 1 and 2 build on refinement of arrays tested in a pilot study engaging children to use animal metaphors as references for physical mapping of small numbers 1 to 4 (Bjørnebye et al., 2017; see Appendix 1 for details). As illustrated by Figure 16, the arrays in the pilot study were extended to a 16-dotted circle (d = 2 m) in the post-task, which in turn was extended to include 50 dots (d = 3 m) in the *Navigation task* in focal study 2 and 100 dots (d = 4 m) in ETP 2. The arguments for these changes were threefold. First, large multi-dotted arrays allow a structural and embodied approach to number-space mappings in three dimensions (cf. DP 1 and 2) that also foster multimodal experiences of cardinality, ordinality and arithmetic (cf. DP 3). Second, I recognised the value of developing designs that enhance shared experiences for learning through imitated action (cf. DP 4). Third, the large-scaled matrices allow the cultivation of mathematical thinking in game, rule, and play-based settings, thereby adding layers of meaning to the number-space associations (cf. DP 2). In addition, the large circles could also function as sites for assessment. For example, the 100-dotted circle in ETP 2 was used as starting point for several games that focused on aspects of the min strategy (e.g., the *Tag a number* activity), and it also demonstrated its usefulness in assessing children's bodily grounding of counting-based addition (cf. the *Min task* in focal study 3).

Figure 16 also shows that the symmetrical structured arrays used in the fixed task in ETP 1 (i.e., the *Imitate animal*-activity) builds on refinements of the designs in the pilot study, and the rationale for this modification was twofold. First, educational research suggests that young children identify canonical (symmetric) patterns faster than asymmetric and linear configurations (e.g., Wender & Rothkegel, 2000). Second, supported by observations in the pilot study and guided by DP 2, I assumed that the use of limbs in the simultaneous tagging of symmetric arrays was more in line with the children's prior experiences regarding body postures than physical adaption to non-symmetric configurations. Accordingly, the fixed tasks used in ETP 1 and 2 also followed the iterative process of re-design, however framed by a larger time interval.

5.2.4 Implementation: Kindergarten teachers and researchers roles and responsibilities

My role as the facilitator during interventions was supported by experiences from mentoring and coaching children in sports (e.g., football, handball), my position as a physical educator and math teacher in primary school and my work as a teacher educator in math, which also involved courses in outdoor embodied learning in various mathematical subjects. I also gained valuable experiences in guiding children in the pilot study (see Appendix 1).

My role as a participant researcher in the project aligns with criticism raised against the illusion that external facilitators' adaption of a passive and technical role reflects neutrality in the research process in participator inquiries (Denzin & Lincoln, 2008). Through my role as a research facilitator, I took active part in leading the joint sessions, in guiding and motivating the children and in collaborating with the KTs to support the implementation of the programmes. The participating KTs responsibility during the fixed sessions was, to their best abilities, to improve flow in activities and to support children's motivation and well-being, and thus improve the conditions for children's learning. When the KTs (or the researcher) felt that the original layout was too difficult for the child to handle (either the fixed or joint activities), the supervisor could provide one-to-one guidance, adaptation, re-modelling and simplify it according to the child's performance level. For example, in the fixed task in ETP 2 (i.e., the *Min task*), simplification could be reflected in setting the dice to small numbers, while joint verbal counting, physical modelling (cf. DP 4) and a helping hand could scaffold the child to master the counting-based addition.

5.3 Participants, samples, and data collection

The two kindergartens (referred to as *Kindergarten 1* and *2*) in this dissertation were strategically chosen due to the conformity between their focus on outdoor pedagogy and my research goals of exploring epistemic processes related to embodied designs, but also for reasons of convenience as volunteer participators (Appendix 3). Individual testing after interventions in these ECEC institutions provided the empirical data for the three focal studies.

With the written consent of parents (see Appendix 4), 3- and 4-year olds (17 children in *Kindergarten 1* and 10 children in *Kindergarten 2*) participated over a period of eight weeks in ETP 1. Seven month later, the same groups participated in the five-week long ETP 2, but implementation in *Kindergarten 1* was only used for testing and refinement. Consequently, only data from *Kindergarten 2* regarding ETP 2 is included in this dissertation (cf. focal study 3).

Table 2 provides an overview of context, interventions and participants, selection criteria, knower-level-data and average level of participation in the interventions, types of test procedures used, and the amount of video material from the individual tests used in the focal studies (the three articles provide more detailed information).

Table 2: Overview of the focal studies, context, tutors, and participants, selection criteria, testing procedures and data material

	Focal study 1	**Focal study 2**	**Focal study 3**
Context	Kindergarten 1 – asphalted area (atrium)	Kindergarten 2 – asphalted area	Kindergarten 2 – asphalted area
Year and period	Autumn 2017. 8 weeks.	Autumn 2017. 8 weeks.	Spring 2018. 5 weeks.
Intervention	ETP 1, 13 sessions	ETP 1, 7 sessions	ETP 2, 7 sessions
Tutors	3 KTs and research-facilitator (me)	1 KT and research-facilitator (me)	1 KT and research-facilitator (me)
Participating children	17	10	10
Selection criteria	Age, ETP 1, knower-level	Age, ETP 1	Age, ETP 2
Sample selected for analysis	8 (4 girls, 4 boys; mean age 4:3, range 3:11-4:9): Two C2- knowers, six C3-knowers	10 (4 girls and 6 boys; mean age 4:2, range 3:10-4:8). Seven CP-knowers, three subset-knowers	10 (4 girls and 6 boys; mean age 4:9, range 4:5-5:3). Nine CP-knowers, one subset-knower
Average participation for sample	9 sessions; 7 h	6 sessions; 6 h	6.4 sessions; 6 h 24 min
Participating children selected for multi-case analyses	6	8	7
Individual testing procedures	- Standardised[9] and modified Give-N task - Navigation task	- Standardised and modified Give-N task - Jumping task	- Give-N task - Min task
Time of video-recorded material from individual testing	120 minutes	200 minutes	150 minutes

[9] Due to autumn vacation, the standard Give-N test in focal study 1 was given two weeks after the modified Give-N test (see description of the Give-N procedure on p. 109 in Article 1 and in Chapter 3.2.2). The purposes of the dual testing were to base the selection from ETP1 in K1 to include extreme cases (i.e., subset-knowers'), and to use the two sets of knower-level data for comparing performance of exact production of small sets in different settings.

In Table 2, it can be seen that the empirical data collected for the focal studies have a qualitative format in the form of video recordings of individual testing. The use of audio-visual materials in research has several advantages (Heath et al., 2010), and includes two main gains for my dissertation. Firstly, the video material enable me to analyse different influential and interrelated dimensions of design, instruction, talk and embodied action, and through the embodied theoretical lens direct my attention to the children's number-space mappings in real time (Knoblauch et al., 2012). Secondly, since action possesses the dual feature of being both context sensitive and context renewing, it is difficult to capture in situ. Consequently, the video-recorded material allows me to recapture the children's speech, use of tools and body-spatial interaction in detail (cf. the framing question of the dissertation).

Data collection of the ETPs include video of the sessions and unstructured log capturing self-reflection and feedback from KTs. A general description of how this empirical material was used for evaluation and refinement of the joint activities is provided in subsection 5.2.3. However, only data from individual tests are included in the focal studies (i.e., 120 min, 200 min and 150 min in focal study 1, 2 and 3, respectively; see Table 2). Although this is a methodological weakness considering the DBR-part of my thesis (see clarification of this limitation in section 2.6), this choice enables me to answer the research questions in the focal studies, but also reduce the data material to a manageable size. In retrospect, however, I acknowledge that a focal study aimed at the DBR-part of my dissertation study could have provided a more rigorous understanding of the role the ETPs played in supporting the young children's grounding of mathematical thinking (see further reflections around the quality of the study in subsection 5.5.5).

As Table 2 shows, the initial selection criteria for children in focal study 1 was, in addition to age and participation in ETP 1, based on their knower-level, and includes the biased sample of C2- and C3-knowers as assessed by the Give-N post-task, while focal study 2 and 3 include all children in the ETPs as samples. To identify prototypical cases as well as maximum variations of task behaviour in these samples (Flyvbjerg, 2006), each trail from every child was included in the initial analyses. Based on a cross-case comparison of this rich and broad empirical material, patterns of grounding mathematical thinking in embodied or simulated action emerged and were analysed using a multi-case methodology.

In Table 2, it can be seen that most children were included in the multi-case analyses (i.e., 6 out of 8, 8 out of 10, and 7 out of 10 in focal study 1, 2 and 3, respectively). The selection criteria for the multi-case analyses were based on the goal of examining recurring patterns and variations within each identified category of task behaviour, and thereby provide an in-depth exploration within the criteria set for being a member of the particular class[10]. As noted by Eisenhart (2009),

> "In striving for theoretical generalization, the selection of a group or site to study is made based on the likelihood that the case will reveal something new and different, and that once this new phenomenon is theorized, additional cases will expose differences or variations that test its generalizability. The criterion for selecting cases from which one will generalize is not random or representative sampling but the extent to which the cases selected are likely to establish, refine, or refute a theory […] the goal of theoretical generalization is to make existing theories more refined and incisive." (Eisenhart, 2009, p. 60)

This is consistent with Yin (2009), who argues that since case studies are generalisable to theoretical propositions and not to populations, the choice of cases in qualitative research should always be based on a specific theory that seeks to be tested (Yin, 2009). Before I will outline the analytical techniques used to achieve generalisation in my dissertation, I will present the testing procedures used in connection to the focal studies.

5.3.1 Procedures of the post-tasks

A researcher led all the testing, one child at a time. The post-testing procedures include the modified *Give-N task*, the *Navigation task*, the *Jumping task* and the *Min task*.

The modified Give-N task (focal study 1 and 2)

Figure 17: Array used in the modified *Give-N* task and the *Navigation task*

Figure 18: "Kangaroo-two" using the legs to tag two dots

The modified *Give-N* task used a circle (d = 3 m) with 50 arbitrarily distributed dots (see Figure 17). Positioned outside the circle, the child was asked: "Can you jump a cock-a-doodle-doo-one/kangaroo-two/monkey-three/frog-four?" (cf. Figure 18). If necessary, the child was guided

[10] Detailed descriptions of the selection criteria for the multi-case analysis is provided in the respectively articles. See also subsection 5.4.1.

via a practice trail. The child was told to say what they jumped. In the case of an unclear or absent articulation, the researcher could ask “What did you jump?”. The questions were asked in random order, and the criterion for proficiency on a level of exact numbering was success in at least two out of three trials (Wynn, 1992). Accordingly, a child could be asked from a minimum of eight (all successes) to a maximum of twelve questions.

The modified *Give-N* task builds on the standardised *Give-N* task and is an extension of the task used in the pilot study (Bjørnebye et al., 2017; Appendix 1), and it differs from the standardised version in several ways. First, the modified version allows full-body spatial interaction on the ground, while the standardised task allows the use of upper-body action in the manipulation of concrete items. Second, in the standardised task, the child is both asked (e.g., “Can you give the puppy two items?”) and supposed to respond verbally by using number words (e.g., “Is that two items?”). In contrast, the modified *Give-N* task uses animal metaphors as verbal references of numerosities (e.g., “Can you jump a kangaroo-two?”), and only requires verbal confirmation if the articulation is unclear or absent. Third, in contrast to the standardised *Give-N* task’s requirement of producing a pile of items that are delivered to a puppy, the modified version is based on the use of body limbs in the coupling of spatial distributed dots on the ground to establish one-to-one correspondence in accordance with the requested cardinal value. Fourth, unlike the modified version, the standardised *Give-N* task follows the titration method, which means that if a child is unable to respond on a number (they are asked in ascending order), then no higher number is asked (see details in subsection 3.2.2). Finally, the modified *Give-N* task is only tested within the subitising range one to four as opposed to the standardised version, which also asks for higher number for assessing abilities in using a counting procedure.

The Navigation Task (focal study 1)

The *Navigation task* used a circle with 50 arbitrarily distributed dots inside and four coloured lines outside (see Figure 17). According to the criteria that each trail started (A) and ended (B) in a pair of coloured lines that intersected at the centre of the circle, there were four ways of using coloured cues to communicate the path of navigation (i.e., blue-orange, orange-blue, white-red and red-white). First, the researcher presented the task: “You are to jump from the red line to the white line (pointing), and you must tell (i.e., use animal metaphors as verbal references) what you jump”. After the completion of a trial, feedback was provided if the child skipped articulation (“remember to say what you jump”) or double-tagged (“do not use the same dot twice"). During action, the experimenter could give hints to remind the child of the aim of the task (e.g. "You are to jump to the red line"), provide encouragement (e.g. "and then") or address unclear articulation (e.g. "You said?"). Each child was asked to produce a minimum of two and a maximum of four trails.

The Jumping Task (focal study 2)

Figure 19: The level 1 task *2+y=3*

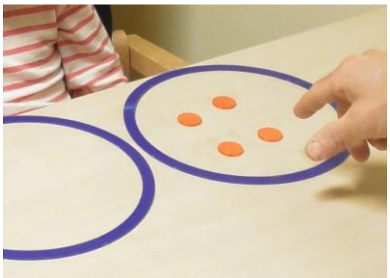

Figure 20: "If you jump a frog-four here…"

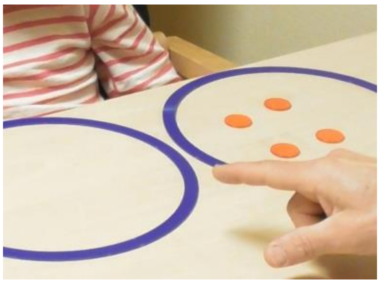

Figure 21: "…and then you jump something here before you jump out…"

The *Jumping task* holds the syntactic form *x* + *y*= *z* and may be categorised as a *Part-Part-Whole*-problem with an unknown part (Sarama & Clements, 2009). The researcher and the participant sat on opposite sides of a table. Two circles (d = 18 cm) formed the boundaries of the parts, and twenty items were available for modelling. The circle to the left of the child contained an array within subitising range 1 to 4 (symmetric patterns for 3 and 4). The tasks were categorised into two levels:

Level 1: The whole (*z*) within subitising range (e.g., *2+y=3;* see Figure 19).

Level 2: The unknown part (*y*) within the subitising range, the whole (*z*) above.

At least two tasks were given at each level, and succession to level 2 was contingent on solving the previous tasks. The task started with the interviewer putting *x* items in the circle to the left of the child. For example on the *4+y=7-* task, the interviewer said "If you jump a frog-four here (pointing at a symmetric 4-dotted array; see Figure 20), and then you jump something here (pointing at the empty circle; see Figure 21) before you jump out (pointing). If you jumped seven in total, what did you jump here then (pointing at the empty circle)?" The interviewer could promote guesses, argumentation and contextualisation via open questions (e.g., What do/How did you think? Can you make a guess/check that out/show? What did you jump?). Closed questions (e.g., Can you count (all)/use items?) were restricted to the first level and only given if the child was unable to respond to the open questions. The task was repeated if the child needed encouragement.

The Min Task (focal study 3)

Figure 22: Two dice and the 100-dotted circle

Figure 23: Role two dice, pick up the largest one

Figure 24: Tag the largest addend outside the big circle

Figure 25: Counting-on inside the circle

Figure 26: Keeping balance while expressing the sum

The *Min task* used two dice and a circle with 100 randomly distributed dots (d = 4 m; see Figure 22). First, the researcher introduced the child to the *Min task*: “Do as we did in the game earlier”, which implicitly meant performing the following four steps:

(1) Roll two dice, compare the values (e.g., 3 and 6; cf. Figure 23) and pick up the one with the smallest value.

(2) Articulate and physically tag the largest addend as a whole (e.g., use the feet to tag the dice and say “six”; cf. Figure 24).

(3) Enter the 100-dotted circle, and informed by the value of the handheld dice, use the feet to tag dots and verbally count-on the number of times equal to the smallest addend (e.g., “seven, eight, nine”; cf. Figure 25).

(4) Keep the balance and articulate the sum in the final tagging (cf. Figure 26).

The researcher then asked: “Can you roll the dice? Which dice should you pick up? What do you do with the other dice?” If necessary, the child was guided via a practice trial. To ensure variation in numbers to add, at least three tasks were given at levels 1 to 3: Level 1 – two dice 1 to 4; Level 2 – two dice 1 to 6; and Level 3 – one dice 1 to 6, the other set to 3, 5 and 6, respectively. During task solution, the interviewer could ask: “What did you get?”, “What do you say/do?” and “How many did you get?”.

5.4 Data analyses and theory building

In contrast to the recognised *Gold Standard* of quantitative research that emphasises on the use of large or random samples and controlled trials to achieve generalisation, the notion of generalisation in case studies is a debated issue as it requires extrapolation that can never be completely justified logically (Firestone, 1993; Gerring, 2007). However, generalisation is intertwined with humans' cognitive abilities and is therefore a fundamental concept for human beings to interact with the world in a coherent way (Robinson & Norris, 2001; Ruddin, 2006). Without generalisation beyond the data, there is consequently no theory or insight and therefore no need to do research (Mintzberg, 2017). In this section, I will outline the analytic approaches and tools used to achieve generalisation in my research study.

5.4.1 Generalisation in the focal studies

The focal studies used principles from the EC framework for analytic generalisation of the empirical material consisting of video of 3-to-5-year-olds' grounding of mathematical thinking in talk and embodied interaction. Basic principles for optimising analytic generalisations in multi-case studies include depth and relevance of the attributes associated with the classification (Kennedy, 1979). To address these aspects in the focal studies, I used the analytic techniques pattern matching, cross-case synthesis and explanation building (Yin, 2009).

In the first step of the analysis, I combined pattern matching logic with a micro-analytic approach for an in-depth exploration of how the task behaviour cohered with relevant attributes of the mathematical targeting domains. Then I conducted a cross-case analysis to identify diversities, gaps and shared patterns. Based on this comparison, general patterns and discrepancies emerged, which I synthesised into categories of grounding mathematical thinking in talk and embodied interaction. Finally, supported by principles of multi-case methodology, I discussed the results from each identified category of task behaviour from the EC perspective, thus providing confirmatory or disconfirmatory evidence shaping the theory (Dooley, 2002). Preliminary results and ongoing analyses were also discussed with other researchers at conferences and in presentations for the field of practice (Bjørnebye, 2019)[11].

[11] CERME11 Conference in 2019, Utrecht, Nederland and POEM4 Conference in 2018, Kristiansand, Norway.

Table 3 provides an overview of theoretical models, main test procedures, data of analysed trials and correct responses, and categories that emerged from the analyses in the focal studies.

Table 3: Overview of theoretical concepts and frameworks, main testing procedures, number of trials examined, frequency of correct responses and categories emerging from the analyses in the focal studies

	Focal study 1	**Focal study 2**	**Focal study 3**
Theoretical frameworks, shared and specific concepts[12]	- Embodied cognition - CMT - Conceptual metaphoric mapping	- Embodied cognition - Embodied numerical cognition - Number-space mappings	- Embodied cognition - Embodied numerical cognition - Number-space mappings
Main testing procedure	Navigation task Modified Give-N task	Jumping task	Min task
Targeting mathematical domain	The idea of cardinality (exact production of small sets)	Parts-whole reasoning	Counting-based addition, the min strategy
Number of trials examined	22 tasks (106 body-spatial couplings)	42 tasks ($x+y=z$; y unknown)	106 additions
Number of correct mappings/ reasoning/ strategies	Coherence in verbalised body-spatial couplings in 96 out of 106 mappings of sets 1 to 4	Valid reasoning in 37 out of 42 tasks	Correct modelling of the min-strategy in 81 out of 106 trials
Findings: Names of categories of task solution identified	1. Inconsistent cross-modal mapping of numerosities 2. *Walkers*: Rigid cross-modal mapping of numerosities, problems in navigation and pattern recognition 3. *Jumpers*: Fluent and coherent cross-modal mapping of numerosities, goal-directed navigation	1. Inconsistent production of valid arguments 2. Patterned counting to support coherence in reasoning 3. Linear touch counting to support coherence in reasoning	1. Preference in retrieving visual information from handheld dice 2. Preference in visuo-tactile representations (cf. touch counting) 3. Preference in mental-based representations (cf. handheld dice) 4. Flexibility in representational modes 5. Incongruence strategy modelling

Table 3 shows that concepts from the EC framework were used as analytical tools to explore the children's grounding of the idea of cardinality, parts-whole reasoning and counting-based addition, and that some were specific tools while others were shared across the focal studies. A principle from the EC framework guiding the development of the analytical tools used in the focal studies was the assumption that the grounding of mathematical thinking was modelled as number-space mappings mediated in modality-specific systems (e.g., kinaesthetic, tactile, auditory, visual-spatial; see the assumptions stated in section 2.5).

[12] See the outline of the theoretical framework in sections 2.2 and 4.2.

Figure 27: "Six" - distinct number-space mappings in focal study 3

Figure 28: "Frog-four" - distinct number-space mappings in focal study 1

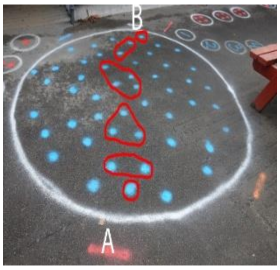

Figure 29: Connected number-space mappings in focal study 1

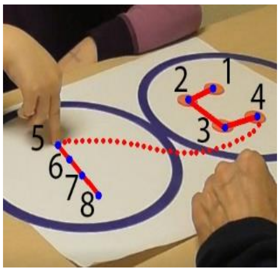

Figure 30: Connected number-space mappings in focal study 2

Based on this and to facilitate the organisation of the empirical material, video of individuals' task behaviour was segmented into modal units (e.g., spoken number words, finger gestures, body postures tagging units or wholes; see Figure 27) as the smallest coding element (Jacobs et al., 1999). In focal study 1, basic coding units were theorised as relational phenomena by the notion of *conceptual metaphorical mapping*[13] (see Table 3). For example, the use of four limbs in verbalised (i.e., "frog-four"; see Figure 28) coupling of a 4-dotted array was modelled as a cross-modal mapping from the visuo-spatial domain to the kinaesthetic and verbal domains (i.e., the directional dimension). Similarly, focal study 2 and 3 used the notion of *number-space mappings* to model connections and directionality among coded elements (see Table 3). An example of number-space associations in focal study 2 includes mappings of the mental image of a frog-gestalt to the spoken word "frog-four" and to a four-fingered square all-at-once gesture in the empty circle (e.g., in guessing "frog-four" in the task $2+y=6$).

To further capture connections in mathematical thinking and its relation to tools and context, I assembled coded segments of distinct number-space mappings into drawn clips (cf. Ekdahl et al., 2016). These accumulated ways of representing several connected number-space mappings were in turn linked to rich descriptions from transcripts. For example, Figure 29 from focal study 1 illustrates a series of clips (i.e., closed curves represent distinct number-space mappings in the form of tagged sets). Similarly, Figure 30 from focal study 2 illustrates a series of connected number-space mappings in the form of a touch counting sequence, which in turn matched the logic of the min strategy on the $4+y=8$ task. Accordingly, the relation between simultaneity and connection was closely associated with congruence in embodied thinking

[13] For consistency in the terminology, I will mainly use the notion of number-space mappings.

since it allowed me to study number-space mappings as both distinct (e.g., discrete motor units such as finger gestures) and relational (e.g., additive reasoning) phenomena.

To give further substance to the analytical generalisations, I emphasised that the children's task behaviour should be observed in several test experiences (i.e., 106 couplings of numbers in focal study 1, and 42 and 106 tasks in focal study 2 and 3, respectively; see Table 3). In Table 3, it can be seen that the analyses of the focal studies identified three or four main categories of task behaviour, each in which reflects characteristic features of grounding mathematical thinking in spatial extensions and locations that go beyond the information provided by the frequency of correct responses as shown in Table 3. Below, I will outline technical aspects concerning the conduct of the coding.

5.4.2 Qualitative data analysis

Qualitative data analysis (i.e., to connect) rests on the distance created by separating the empirical materials into smaller segments for consideration, reflection and interpretation (Ellingson, 2011). Accordingly, the credibility of the qualitative data analysis in this dissertation study is reflected in how I managed multiple dimensions of tensions between the data and the codes to ensure reconstruction of evidence in terms of tracing back to the original material and related codes (Chenail, 2012). To maintain a chain of evidence, I created a database containing information of each participant. Guided by the framing question of my dissertation, the coding structure aimed to capture the bidirectional, multimodal, distinct and relational nature of number-space mappings, and its coherence to the targeted mathematical domains. In order to support the dissertation's objective of in-depth exploration of children's grounding of mathematical thinking in embodied action, I combined clips of number-space mappings and transcripts of verbal accounts of the child's (and the interviewer's) mathematical thinking to provide accurate, detailed and contextualised descriptions of the data. In addition, crutch words, sighs and non-verbal cues (e.g., pauses, overlaps, pitch, volume and fluency) can convey important aspects of mathematical thinking. Consequently, the trustworthiness of the interpretation of the transcription rests on the inclusion of crucial pauses and overlaps in the video recorded material (Silverman, 2014). To address this tension between data and codes and to provide a basis for examining fluency and connection of number-space mappings, the transcripts provided accurate information of pauses (e.g., "pauses in 12 seconds"; cf. focal study

2) or used brackets for indicating halts for thinking and reflection (e.g., "[Hesitates] Hmm [pauses]…": cf. focal study 2).

The code structure emerged as a blend of a priori categories and new categories derived from a grounding approach. The deductive dimension was mainly influenced by the EC framework and the research questions asked, where the assumptions made enabled logical inference of codes associated with coherence in the situating of mathematical thinking in sensory-motor experiences. In focal study 2, for example, it was assumed that coherence in thinking was mediated verbally and non-verbally in a way that matched the logic of additive reasoning, and that signatures of simulated action involved verbal use of animal metaphors (e.g., "frog-four"), and gestures and modelling matching the canonical structures of the training-based experiences from ETP 1. Based on these assumptions, verbal use of animal metaphors, all-at-once gestures, canonically structured counting and modelling were derived as a priori codes, as were the mathematical building blocks of additive reasoning (i.e., the more-and-less relation, the successor rule and the counting principles of Gelman and Gallistel, 1978).

Inductive based codes came primarily from two types of sources. Firstly, observations during the intervention provided knowledge about the children's expected behaviour in the post-tasks (i.e., *Navigation task*, *Jumping task* and *Min task*). This was particularly apparent in focal study 3, where the fixed activity and the *Min task* were similar and the piloting of ETP 2 in *Kindergarten 1* provided additional knowledge of expected codes. Secondly, new code categories also emerged during the analyses. This was especially evident in focal study 2 where the mathematical content (i.e., exact production of small sets vs. parts-whole reasoning) and the context (i.e., outdoor bodily experiences in ETP 1 vs. indoor testing) were different (e.g., linear counting emerged as a new category). Furthermore, the introduction of navigation and the 50-dotted circle as novel elements in the *Navigation task* in focal study 1 produced new codes involving cognitive conflicts, failures in cross-modal mappings (e.g., miss-match in the one-to-one mapping of limbs and dots) and the degree of fluency in the physical couplings (i.e., *Walkers* vs. *Jumpers*; see Table 3). However, in light of the complex, bidirectional and multimodal nature of number-space mappings, the code structures reflected only a small portion of possible ways of grounding mathematical thinking in embodied interaction.

5.5 Reflections of the quality of the research

In this section, I will discuss issues related to the scientific rigour and quality of my research, including trustworthiness, reliability and ecological validity, and considerations around methodological choices.

5.5.1 Trustworthiness

A threat to credibility in qualitative research is *confirmation bias* (also referred to as *anecdotalism*), which means that the researcher tends to interpret data based on previous findings or to confirm stated hypotheses (Silverman, 2013). In this regard, there was data in the three empirical datasets of this dissertation that did not cohere with other findings (e.g., the linear counters in focal study 2). However, the explorative approach allowed me to include novel and unexpected findings. In addition, the use of modal units as the smallest coding segment enabled me to provide rich descriptions of distinct and connected number-space mappings reflecting the children's embodied grounding of mathematical thinking. However, a different coding element may have provided other results. Furthermore, researchers with different beliefs, expectations and values than me (e.g., post-humanistic researchers; cf. reflections in subsection 5.6.3 below) may also be able to see and evaluate other things and to identify new patterns in the data material (Silverman, 2013). Based on these reflections, and to counteract confirmation bias and tunnel vision, I addressed several issues regarding transparency and credibility in the data selection.

Firstly, to ensure that the coding was not based on extreme and deviant cases or anomalous situations, all children's task behaviour of the samples were coded. To increase the validity in terms of selecting data in a transparent way, the inclusion criteria of the multi-case analyses were explained in detail (see *Article 1*, *2* and *3*), and the selected cases and data provided illustrative examples and rich descriptions of two or more children from each of the identified patterns of task behaviour (see Table 2). Additional ways of counteracting confirmation bias include triangulation and respondent validation (Silverman, 2013).

Different types of triangulation include data triangulation, theory triangulation, methodological triangulation and investigator triangulation (Patton, 2002). In this dissertation, theory triangulation was achieved mainly through the EC perspective, which draws on multiple frameworks that share a set of foundational assumptions (e.g., CMT in focal study 1 and Embodied Numerical Cognition in focal study 2 and 3; see Table 3). In addition, the focal studies were informed by educational research and models (e.g., empirical research on subitising, addition, reasoning; see Table 3 and Chapter 3), contributing to the validity of my findings. Data triangulation was achieved with several sources of data (e.g., data from standardised and modified Give-N task, *Navigation task*, *Jumping task* and *Min task*), while methodological triangulation was reflected in the blending of principles from DBR and multi-case research and in the use of different analytic approaches to make sound causal inferences and explanations (Cohen et al., 2011). Investor triangulation was achieved through the collaboration with my supervisor in the different phases of the research process. In terms of credibility, this collaboration was particularly valuable in comparing the coding and analysis of the data for common confirmation and understanding (Thurmond, 2001). In addition, I discussed my findings with critical colleagues in the scientific community, thereby challenging my interpretations.

Response validation suggests that the researchers should make multiple efforts to refine tentative interpretations and results based on the subjects' previous behavior (Silverman, 2013). I made several efforts to enhance this aspect of trustworthiness in my dissertation study. Firstly, the implementation of the ETPs prior to post-testing ensured children's involvement in a shared base of exercises targeting specific mathematical domains. Accordingly, information from the children's performances helped me to design testing procedures in accordance with their conjectured skills. Furthermore, respondent validation was integrated in the post-test procedures, so I could validate and refine responses across trials. For example, in focal study 2 and 3, the children were engaged in several levels of tasks that reflected varying degrees of difficulty, so that strategy consolidation and refinement across tasks were explored. The interviewer could also ask open-ended or closed questions to encourage participants to refine, validate and make their thinking explicit.

A final aspect of trustworthiness concerns my close involvement in the various phases of the research process (Barab & Squire, 2004), and the main issue in this regard was that I as a researcher-facilitator was part of the investigated world and featured as an important tool of research, implying a risk of misinterpretations and personal and non-objective perspectives (Giddens, 1979). Although some proponents of qualitative methods argue that the researcher's insight and deep understanding of the context make them the best research tool, despite their biases (Onwuegbuzie & Leech, 2007), I used several strategies to counteract confirmation bias, several of which I have outlined in detail above. Furthermore, the premise of the partnership was that the KTs would only play a supporting role in testing and guidance, as they were not trained in doing design-based research and were too busy for a deeper commitment that involved evaluation and re-design. Nevertheless, their support and knowledge of the children's culture, behaviour and individual preferences made a significant contribution to the dissertation.

5.5.2 Reliability

Reliability in qualitative research refers to the stability and consistency of the findings, meaning that other investigators should be able to examine and replicate my project and come to similar conclusions (Altheide & Johnson, 1994). Accordingly, issues of reliability play a central role in all stages of a qualitative research process, and an important way to increase the consistency of the findings is to document the succession of design, data collection, coding, analysis and interpretation (Ali & Yusof, 2011).

In this dissertation, several efforts were made to support reliability. Firstly, the reliability was increased by the independent analysis and coding by my supervisor and I before negotiating to reach intercoder agreement (Lincoln & Guba, 1985). Secondly, thick descriptions of the two ETPs are included in this dissertation (see subsection 5.2.1), which in sum reported on influences from context, instruction and training before the testing that provided the empirical material of the focal studies. In turn, this also contributes to the replication of the study and it conveys information to KTs in ECEC institutions about practical use of the ETPs. Thirdly, rigid procedural descriptions of the conduct of the post-tasks (i.e., The *Navigation task*, the *Jumping task*, and the *Min task*) can further enhance replication (see details in section 5.3.1 and in the *Articles 1-3*). Finally, detailed and contextualised data excerpts in the analyses, aimed to provide accurate descriptions of the participant's talk and number-space mappings (cf. Silverman, 2013). However, when the children were engaged in the post-tasks and in the ETPs, there was a network of interrelations between variables that constantly changed character through complex feedback loops (Collins et al., 2004). Therefore, the notion of capturing all the influencing variables in my dissertation is hard to justify. However, when doing educational DBR, the impact of external and contextual factors are important for understanding the phenomena examined, and if the ETPs and the post-tasks are not robust enough to adapt to changing conditions, they may not work in real life ECEC contexts. This draws my attention to the practical usefulness of the results of my dissertation.

5.5.3 Ecological validity

The ecological validity rests on the extent to which an inquiry is contextualised in the environment that the phenomenon occurs, and this aspect of credibility was supported by several factors in my dissertation.

Firstly, the embodied designs were situated in frequently used outdoor areas (see Table 2; cf. also section 2.3). This concurs with Barab and Squire (2004), who suggest that in order to gain a more genuine and realistic understanding of learning and instruction, DBR must include naturally occurring contexts and testing scenes in addition to more favourable and constructed settings. In addition, the ETPs were framed within two ECEC institutions with several similarities, but also some differences. In this regard, I consider any differences as a strength to the embodied design's adaptability, which supports the claim that contextual nuances are not decisive for the results as the focus is on young children's grounding of mathematical thinking in motor behaviour shared by children around the world. Together with the detailed descriptions of the embodied activities (cf. section 5.2) and the results of the focal studies, the findings can contribute to a deeper understanding of early body-based mathematical thinking for other researchers and to KTs with different contextual settings.

Secondly, concerning the tools, materials and task structures (cf. Sandoval, 2014), the main part of each session was integrated into an everyday construction activity, and the artefacts and material used (i.e., building bricks, toy-figures, and dice) are elements taken from children's culture of play (cf. DP 2). The guidance structure of the embodied designs also aimed to reflect normal practice, as the children could receive help and support from adults (i.e., KT and researcher) and peers, and they were allowed to regulate their own involvement to a certain extent (cf. the construction activity as part of the fixed tasks; see subsection 5.2.1).

Thirdly, the composite movement-based elements of the activities build on and integrate children's motor skills (e.g., gait, jumping, balancing), which form the basis for meaningful play in ecologically rich contexts (cf. DP 2). Furthermore, the physical imitation of animal behaviour in ETP 1 (cf. the fixed task) is part of children's role-playing (cf. DP 4).

Finally, both participating ECEC institutions integrated the embodied designs in daily practice after the end of the intervention. This is consistent with Brown (1992), who argues that the effectiveness of any DBR-based intervention rests on how applicable and transferrable the produced practical knowledge is to an average pedagogical practice. Put together, these reflections and arguments support the likelihood that the results of this dissertation can be useful outside its own context.

5.5.4 Reflections on methodological choices and limitations

There are several advantages of using the same methodological and theoretical framework in the focal studies. Firstly, the replication provided me as a researcher in-depth knowledge of the underlying assumptions of the EC framework and case-based methodology. Secondly, it allowed me to develop skills in how I could integrate and adapt the specific methods to the research questions asked, which also required technical abilities to perform the analyses (cf. subsection 5.4.2). Thirdly, the qualitatively methodological approach provided the benefit of a flexible and in-depth approach for the examination of young children's grounding of mathematical thinking in embodied interaction (Driscoll et al., 2007).

However, compared to a broader methodological perspective of the studied phenomenon, the selected methodological approach may imply that salient aspects of the research questions asked may have remained unexploited. In addition, studies targeting instruction, facilitation or methodological considerations as phenomena of inquiry may have provided a more rigorous and trustworthy understanding of how and why the ETPs may support children's learning of mathematics. Furthermore, greater practitioner involvement from KTs (e.g., a collaborative learning project) could have increased the relevance of the ETPs for practice (Plomp, 2013). Finally, methodological approaches with controlled training conditions (e.g., one-to-one training), and use of a control group, randomised crossover design (e.g., Dackermann et al., 2016) and larger samples may have yielded more rigorous scientific results.

Further weaknesses of the study concern the design of the activities, which may reflect a biased view of the participating children's motivation and previous experiences. For example, in ETP 1, I acknowledge that the choice of more culturally appropriate themes than animal behaviour may have provided other results (see further reflections in *Article 2 and* section 7.7). Furthermore, the body presents limitations in gesturing larger numbers than those being addressed in this dissertation study. Finally, the wrapping of the post-tests (i.e., the *Navigation task*, the *Jumping task* and the *Min task*) in collaborative contexts and in explorative open-ended tasks may have provided more reliable results in terms of revealing how young children actually use their bodies to ground mathematical thinking onto spatial extensions and locations in ecological rich outdoor settings.

5.6 Ethical deliberations

The dynamic nature of participatory research in educational settings entails the integration of ethical deliberations for good research practice in all phases of the study. To ensure that ethical values were followed thoroughly, I applied the guidelines for research ethics given by *The National Committee for Research Ethics in the Social Sciences and the Humanities* (NESH, 2016). However, as noted by Bertrand Russell, "Ethics is in origin the art of recommending to others the sacrifices required for co-operation with oneself" (Russell, 1917, p. 108). Accordingly, ethical deliberations in research are not just a matter of following strict, predefined standards and regulations. Rather, the nature of ethical concerns in participatory research is complex as it involves observation, intervention and interaction with the research subjects; it is described as co-operative inquiry *with* rather than *on* people (Heron & Reason, 2006). Based on this, this section comprises reflections on how I dealt with ethical issues that emerged in the various phases of my dissertation.

5.6.1 Research ethics

The *Norwegian Centre for Research Data* approved the project (Appendix 2), and the leaders of the two participating ECEC institutions, the KTs and the parents of the participating children gave informed consent (Appendix 3 and 4). Ethical issues were discussed with the KTs, who were also responsible for informing the parents and for recruiting participants. The letter of consent implemented several ethical principles, including issues of confidentiality, anonymity and integrity, storage and processing of data (Appendix 4).

In collaboration with the KTs and guided by the principle of avoiding exposing the children to unpleasant situations related to failures or shortcomings, I tried to interpret and solve emerging ethical tensions in situ. For example, when spontaneously expressed corrections from other children occurred, I emphasised drawing attention to what the children had mastered, which was easy considering that the body most often conveyed some meaningful aspect of the targeting mathematical domain. Later, children could receive one-to-one guidance without interruption from peers. The issue of integrity also arose when the children became tired and reluctant to participate in outdoor sessions (sometimes under the joint activities in *Kindergarten 1*). To address this, the KT (or the researcher) comforted and talked to the displeased children, who could choose to either re-join the group or rest until the next activity.

5.6.2 Confidentiality and anonymity

Guided by ethical principles provided by NESH (2016), the processing of personal data was limited to gender, average age and age range of participants. Related to this, a main issue concerning the framing question of the dissertation was to provide rich and reliable descriptions of the participant's contextualised sensory-motor experiences. However, this pursuit for accurate and in-depth accounts of situated body-spatial interaction made it harder to adhere to ethical aspects such as non-traceability and anonymity (Cohen et al., 2011). To safeguard the children's integrity, picture filters were used to anonymise the children, but clips of sprayed arrays on the asphalt made it possible to trace back the location of the two kindergartens. However, the participating ECEC institutions refused to remove the arrays, arguing that they wanted to use them in their daily practice after intervention. Although this represents an ethical dilemma and weakness in this type of inquiry (Banegas & Villacañas de Castro, 2015), rich descriptions of the study and the use of the designs after the intervention were deemed more important than keeping the identity of the participating ECEC institutions unknown.

5.6.3 On stigmatisation and voice

Although a key principle of research is to avoid stigmatisation of vulnerable groups (NESH, 2016), this emerged as an issue in focal study 1 where the sampling criteria was based on knower-level-data, including only children with immature skills in verbal counting for exact numbering. However, contrary to stigmatisation, the focus was on highlighting subset-knowers' actual abilities in bodily mapping of small sets as a group. This argument was backed up by the results, which showed that children whose performances were labelled *weak* by standardised measures (cf. the Give-N task), showed skills in other modalities for exact production of small sets. Accordingly, focal study 1 demonstrated that a conceptual layer of cardinality could entail new ways of moving and interacting, an important finding for children with preferences in non-verbal approaches to mathematical sense-making. In addition, based on the large proportion of subset-knowers in *Kindergarten 1* (cf. focal study 1), I decided to use the implementation of ETP 2 in *Kindergarten 1* for piloting and testing and to focus on refining the joint activities, thereby allowing the children to experience mastery and meaning rather than becoming despondent due to possible failures and shortcomings.

Another ethical deliberation was dealing with children whose parents had not given consent (Strike, 2006). To meet this ongoing ethical dilemma and to reduce the risk of stigmatisation, the interventions in *Kindergarten 2* used an area nearby (i.e., the end of a parking area 100 meter from the institution) out of sight of non-participating children. *Kindergarten 1* used an atrium that was not part of the outdoor playground. However, about half of the children had to pass the atrium when they walked to the playground, and sometimes they asked to participate. Therefore, a compromise was made to allow non-participating children to use the sprayed arrays for play outside intervention sessions.

A major privilege when conducting and reporting research with children is the opportunity to give a voice to those with less power (Mockler, 2014). However, in practice, I experienced this as a challenging task since the most significant aspects of the children's voice in my dissertation study was in the form of non-verbal representations. Also, from a post-humanistic perspective of early mathematical development, criticism might be raised of the study's way of objectifying the children, claiming that the goal-directed embodied approach hindered the children's feelings and meanings to come to fore. Several arguments underline, however, that the children's voice through bodily action fostered mastery and meaning to their lived reality. Firstly, through the integration of motor resources not usually appreciated in early learning of mathematics, the intervention provided a steep learning trajectory for all children across competencies. Secondly, the design's inclusion of music, rhythm, play, construction, collaboration and composite bodily movements are ecological elements that conflate with children's daily practice (see subsection 5.2.1). Thirdly, and most importantly, I think, was the overall positive feedback from children (e.g., smiles, laughter, motor engagement and excitement). For example, when I arrived at the sessions, the usual scene was children shouting and gesturing as they ran towards me: "Math-Morten is here, come on everyone! What are we going to do today?" Together with positive feedback from KTs, leaders of the ECEC institutions and parents, these aspects resulted in a prolonged partnership in the form of examining new embodied designs. However, due to the Covid-19 pandemic, the planned research programmes are temporarily postponed.

6. Summaries of the articles

This chapter provides brief summaries of the three articles included in this dissertation. The articles present the empirical findings that constitute my dissertation. Combined, the findings reported in these articles comprise young children's grounding of the ideas of cardinality and addition in embodied interaction, and the simulation of bodily experiences of numerosity in reasoning about parts-whole relations. A further discussion of the results follows in Chapter 7. The two first articles were co-authored. The author's declaration in Appendix 5 provides detailed explanations of individual contributions to the articles.

6.1 Article 1 (focal study 1)

Bjørnebye, M., & Sigurjonsson, T. (2020). Young Children's Cross-Domain Mapping of Numerosity in Path Navigation. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics Education in the Early Years: Results from the POEM4 Conference, 2018* (pp. 109-126). Cham: Springer International Publishing.

There are three main arguments for designing embodied training programmes based on innate abilities of numerosities in the form of a structured approach to exact production of small sets for young children with partial concepts of cardinality. Firstly, the ability of exact enumeration is fundamental to later development of fluency in arithmetic. Secondly, the cultivation of number sense and culturally acquired knowledge of numbers is particularly important for children who are struggling to master exact numbering in standardised tests. Thirdly, research has shown that full-body movement provides multimodal cues to represent and retrieve numerical knowledge. Based on this, the multi-case study reported in *Article 1* builds on eight 3-to-4-year-old subset-knowers' (C2- and C3-knowers) participation in ETP 1 conducted outdoors. The interventional activities involved the grounding of the idea of cardinality in physical couplings of 1-to-4-dotted canonical (symmetric) structured arrays on the asphalt using the animal metaphors *cock-a-doodle-doo-one*, *kangaroo-two*, *monkey-three* and *frog-four* as verbal references (e.g., saying "kangaroo-two" while simulating kangaroo behaviour using the legs to tag two dots).

Individual post-testing in the *Navigation task* collected empirical data to analyse subset-knowers' abilities to transform and adapt the canonical structured embodied experiences to a novel context of 50 randomly distributed dots, using the four animal metaphors from the intervention as verbal references for physical mapping of arrays of dots while moving and maintaining a course across the circle. The following research question was explored: "What inhibits and scaffolds C2- or C3-knowers' mapping of spatial structured knowledge of numerosity across conceptual domains in a navigation task?"

The notion of *conceptual metaphorical mapping* theorised the grounding of the idea of cardinality in sensory-motor experiences, focusing on similarities and differences in cross-domain mapping of numerosities (e.g., mental magnitudes, and kinaesthetic, verbal, and body-spatial modalities). Based on an interpretive stance adapting principles from multi-case methodology, the analyses showed that the subset-knowers possessed the ability to use animal metaphors as verbal references for bodily production of sequences of small sets (e.g., "monkey-three, kangaroo-two, frog-four" denoted as *3+2+4*) in a manner that exceeded their cardinal-knower level as assessed by the standardised Give-N task. Furthermore, the analyses demonstrated quality differences in goal-directedness, abilities in visual pattern recognition and verbal retrieval of animal metaphors, which in turn seemed to influence the degree of fluency and abilities in adapting the body posture to different arrangements of dots. Three categories of task behaviour were identified: (1) Children who failed in some of the cross-modal mapping of numerosities; (2) Children showing problems in navigation and pattern recognition (*Walkers*); and (3) Children who showed coherence in cross-domain mapping of numerosities and goal-directed navigation and appropriation of the spatial structured affordances (*Jumpers*).

The production of small sets for children opting to walk seemed to be negatively influenced by a rigid idea of how the configuration of potential arrays should look like (e.g., 2-dotted arrays had to appear in parallel to their visual field, and 3- and 4-dotted arrays had to be regular shaped). This suggests that the learnt (cf. ETP 1) and the innate (cf. subitising and OTS) capacities for recognising small numbers worked as parallel processes. In contrast, children opting to jump showed coherence in metaphor usage, body-spatial coupling and navigation, suggesting that the integration of learnt and inborn spatial mental models to process numbers supported a rapid production of sequences of small sets.

The standardised Give-N task requires the child to produce a requested number of tangible objects (e.g., "Can you give the puppy three items?"; see section 3.2.2). In contrast, the *Navigation task*, by reversing the mapping order (i.e., "Can you tell us what you jump?") in a context promoting autonomy (i.e., free choice of sets), authentic movement patterns and use of metaphors that convey spatial structured information of numbers, showed that subset-knowers' cardinal-knower-level behaviour can reflect complementary layers of meaning (both subjective and objective) to the idea of cardinality. To conclude, based on the EC framework, this study adds to the debate on how to use the body in early learning of mathematics as it reveals characteristic features, similarities and diversities in young children's grounding of the idea of cardinality in physical interaction in arrays on the ground. The findings also contributes to the knower-level theory as it shows that subset-knowers' concepts of cardinality in terms of wholeness of sets are more diverse than what data from the standardised Give-N task can indicate.

6.2 Article 2 (focal study 2)

Bjørnebye, M., & Sigurjonsson, T. (Submitted). Young children's simulated action in additive reasoning

The multi-case study reported in *Article 2* provides an in-depth exploration of young children's simulated action in a parts-whole task (named the *Jumping task*) from the perspective of EC. The motivation for conducting the research is threefold. Firstly, contemporary research emphasises the importance of early development of reasoning about numerical relations. Secondly, recent research that posits a close relationship with spatial and numerical domains shows that children develop skills in arithmetic reasoning at earlier stages than previously suggested. Thirdly, two lines of research of the embodied perspective focus on the role of full-body and upper-body spatial interaction, respectively, in the creation of number-space association. Yet, little is known about how the blending of these forms of embodied numerosities in the form of simulated action might support additive reasoning.

Informed by theory of embodied cognition, the study assumed that sensory-motor experiences in the form of multimodal simulation matching a targeting domain might enrich encoding and thereby facilitate the re-enactment in simulations of these ideas. The context of the *Jumping task* is based on ETP 1 engaging ten 3- and 4-year-olds in verbalised body-spatial mapping of 1- to 4-dotted canonical (i.e., symmetric) patterned arrays on the ground (e.g., saying "monkey-three" using both legs and one hand to tag three dots).

Post-data showed that the ten participants mastered exact production of small sets 1 - 4 in the modified large-scale Give-N task in a 50-dotted circle (e.g., "Can you jump a monkey-three?"; see Figure 16), while results from the standardised Give-N task showed that seven children were cardinal principle knowers and three were subset-knowers.

Based on empirical material from individual post-testing with the *Jumping task*, the study reported in *Article 2* examined the children's ability to re-enact the spatially structured outdoor experiences of numbers in an indoor setting using finger gestures and manipulation with tools to simulate the full-body action while targeting additive reasoning as a new content domain. The specific research question was: "What characterises 3- and 4-year-olds' talk, use of tools, gestures and simulated action in reasoning about parts-whole relations?" The aim of the study was to develop our understanding of promoting and inhibiting characteristics related to young children's use of simulated action to bridge different forms of bodily knowledge of numbers in more complex arithmetic reasoning.

The pattern matching and cross-case comparison identified three categories of task behaviour in the *Jumping task*: (1) Subset-knowers with partial and inconsistent abilities in reasoning; (2) CP-knowers that supported additive reasoning in the re-enactment of the canonical patterns from the interventional experiences in the form of patterned counting; and (3) CP-knowers opting for linear touch counting of the unknown part to support coherence in the parts-whole reasoning. The main characteristics of the first two groups comprise partial and comprehensive abilities to re-enact and integrate the canonical structured sensory-motor experiences in the additive reasoning, respectively. In contrast, children who opted for linear modelling reflected a high degree of independence from the interventional experiences. These diversities in strategy usage concur with the assumption of EC holding that individuals can ground abstract concepts in multiple and mutually constitutive simulations, and that these simulations can be partial and incomplete (see section 4.2 for details).

Based on the EC framework, this study adds in several ways to the field of educational research focusing on early abilities in parts-whole reasoning. Firstly, the findings show characteristic and deviant features in how young children might re-enact full-body number-space mappings in simulated action in a way that supports congruence in additive reasoning in a realistic parts-whole task. Secondly, it demonstrates patterns in young children's off-loading of mathematical thinking onto spatial structures, gestures and tools. Finally, the study highlights the relation between spatial structured full- and upper-body movements in the early learning of mathematics.

6.3 Article 3 (focal study 3)

Bjørnebye, M. (Under review). Full-body interaction in young children's modelling of counting-based addition

The transition from counting-all to counting-on strategies involves a conceptual leap in the early learning of addition. To address this change and motivated by the EC framework's assumption that many everyday experiences are closely related to basic arithmetic, ten kindergarteners participated in a guided outdoor intervention (i.e., ETP 2) that involved grounding of counting-based addition in talk and embodied interaction. Contextualised by a 100-dotted circle (d = 4 m) using two dice, experiences with the min strategy involved picking up the smallest dice and physically tagging the largest dice as a whole outside the circle (e.g., in *2+4*; say "four" while stamping next to the largest dice). Then, inside the circle with the handheld dice, the child should walk and tag dots the number of times equal to the smallest addend to produce the sum (e.g., "five, six"). Based on these interventional experiences (named the *Min task*), the research question guiding the multi-case inquiry reported in *Article 3* was: "What characterises 4- and 5-year-olds' talk, use of tools and full-body-spatial interaction in the modelling of the min strategy in a 100-dotted circle?" With a focus on examining deviant and recurring patterns in how young children's talk and physical spatial interaction can match thinking about addition, the research aimed to deepen the understanding of how outdoor embodied experiences can facilitate meaningful learning in early years.

Individual post-testing with the *Min task* produced empirical data consisting of 106 tasks (e.g., *3+2*, *5+5*, *6+3*). The overall results showed that two participants had inconsistent proficiencies in strategy modelling, while eight children demonstrated task behaviour that matched the min strategy. The comparison showed that main differences in strategy modelling was reflected in the way the children off-loaded the additive thinking in the counting-on part of the min strategy (i.e., task behaviour inside the circle that reflected the ordinal part of the counting-based addition). The following four categories of task behaviour was identified: (1) Preference in retrieving visual information from handheld dice; (2) Preference in use of visuo-tactile representations of numbers (i.e., touch counting); (3) Preference in mental-based retrieval of numerical information from handheld dice; and (4) Varied use of the above forms of representational modes. A key feature for children using mental representations was a fluent integration with the verbal and bodily modalities, reflected in louder articulation and harder physical tagging of the largest addend and the sum, rhythmic mapping of the smallest addend and bodily rotations in expressing the sum. In contrast, characteristic task behaviour for children who perceived visual and tactile information from the handheld dice to guide the ordinal structure included stiff body-spatial couplings (i.e., gait) and a monotone speech. Despite differences in strategy usage, the general results suggest that sensory-motor experiences might support young children's modelling of counting-based addition. For KTs, the identified characteristics associated with efficiency and congruence can be used as guiding cues for the fostering of fluency in children's full-body modelling of counting-based addition.

Based on the EC framework, this study contributes in several ways to educational research that focuses on the moving and acting body in early learning of arithmetic. Firstly, it demonstrates deviant and recurring patterns in young children's grounding and off-loading of additive thinking in sensory-motor experiences, gestures, tools and spatial structured affordances. Secondly, the findings show how young children's full-body movement and spatial interaction might match the logic and rules of counting-based addition. Finally, and equally important, the study reveals the potential for integrating expressive movement patterns (e.g., force, tempo, rhythm, rotations) into the early learning of addition.

7. Synthesis of findings, discussion, contributions and summary

The overarching aim of my thesis was to investigate characteristics of young children's grounding of mathematical thinking in sensory-motor experiences, while a sub-goal was to develop knowledge about how embodied designs can facilitate such experiences. Supported by my theoretical perspective, I identified the following four interrelated and partly overlapping aspects of the framing question (cf. section 2.6):

(i) Coherence in number-space mappings
(ii) The partial, situated, bidirectional, distinct and relational nature of number-space mappings
(iii) Number-space mappings in simulated action and off-loading of thinking
(iv) Efficiency in number-space mappings

Structured by these themes and through the lenses of EC, this chapter relates the case-based findings in the three focal studies to previous research. Based on this discussion, and as a conclusion of the research questions asked, I next propose the conjectures of progression paths that along with the ETPs and assessment tools in this research project might be used by KTs to identify, support and cultivate children's embodied action into coherent mathematical thinking. This chapter also includes reflections on the dissertation study's contributions, implications for practice, study limitations and suggestions for further research.

7.1 Levels of coherence in young children's number-space mappings

What characterises coherence in children's grounding of mathematical thinking in sensory-motor experiences and simulated action?

In order to address this issue, three focal studies revealed characteristics and deviant features in young children's attempts to establish coherence in the mapping of mathematical thinking in full-body and upper body movement, spatial extensions, and manipulation of tools (cf. Patro et al., 2014), thereby adding to the physical grounding project of EC (see Anderson, 2003). The mathematical targeting domain's cardinality, addition and parts-whole reasoning constitute building blocks in young children's learning trajectory of arithmetic (Sarama & Clements, 2009). Accordingly, the findings concur with previous studies showing that the different features of numerical relations (e.g., subitising, addition, more-and-less and parts-whole relations) are connected and can be developed in conjunction (Jung, 2011).

In focal study 1, the subset-knowers showed abilities to use animal metaphors as verbal references for exact body-spatial production of small sets in a manner that surpassed their cardinal-knower-level (cf. the Give-N task) while moving across a 50-dotted circle (*Article 1*). In focal study 2, several of the 3- to 4-year-olds' demonstrated abilities to re-enact the canonical structured sensory-motor experiences from ETP 1 to support parts-whole reasoning of numbers (*Article 2*). In focal study 3, most of the 4- and 5-year-olds' showed abilities in synchronising talk, manipulation with two dice and physical interaction in a 100-dotted circle in a manner that cohere with counting-based addition (*Article 3*). According to the suggested learning trajectory of Sarama and Clements (2009), 4- to 5-year-olds' are (in average) able to solve missing addend problems (e.g., 3 + _ = 5) and 5- to 6-year-olds' are (in average) able to master counting-on strategies. Although it is difficult to compare, this (slight) age-discrepancy combined with the relative high frequency of solution (cf. the 3- to 4-year olds' reasoning abilities and the 4- and 5-year olds' modelling abilities in counting-based addition reported in *Article 2* and *3*, respectively; see Table 3) gives reason to ask if the bodily experiences lowered the threshold for establishing coherence with the targeted mathematical domains. General support for this line of reasoning comes from research highlighting embodied simulation, imitation and role-playing (cf. DP 4) as fundamental to human learning (e.g., Ando et al., 2015; Barsalou, Niedenthal, et al., 2003; Donald, 2005; Gallese & Sinigaglia, 2011; Hurley & Chater, 2005). However, the in-depth analyses of the three sets of empirical data showed that the children's task behaviour was not a matter of right or wrong, but rather placed on a continuous scale reflecting degree of concurrence with the targeting mathematical domains (cf. the taxonomy of embodiment in educational settings of Johnson-Glenberg et al., 2014). This is consistent with what Dowker (2005) refers to as a 'zone of partial knowledge and understanding', as she argues that the use of the 'know' and 'not know' dichotomy is insufficient to describe children's arithmetic skills.

As a main observation across the focal studies (cf. *Articles 1-3*), the embodied interaction matched at least partially the targeting content, but the relation to the verbal modality did not always cohere. For example, two children in focal study 3 were unable to match the embodied action with spoken words according to the logic and rules of addition, but their full-bodily movement trajectories and manipulation with tools modelled salient non-verbal parts of the min strategy (*Article 3*). In focal study 2, the subset-knowers showed abilities in re-enacting spatially structured physical experiences in several stages of task exploration (e.g., all-at-once gestures in stating guesses and canonical structured modelling and testing), but they were unable to connect these signatures of simulated action to coherent verbal-driven parts-whole reasoning (*Article 2*). Further examples of action-speech mismatches come from focal study 1 where most errors (i.e., 8 out of 10) were related to the verbal modality (e.g., the use of *cock-a-doodle-doo-one* as verbal referent in the physical coupling of 3-dotted arrays; *Article 1*). In over ninety percent of the identified sets of one to four limbs that touched the asphalt, there was coherence across the verbal and body-spatial modalities, while only two of 106 observations showed discrepancies between bodily and spatial domains (i.e., units in the form of limbs touching the ground but not dots). This is consistent with the study by Gunderson et al. (2015), which showed that subset-knowers were more than twice as accurate when labelling sets of 2 and 3 items with gestures than with number words, and especially if the values were above their knower level. Such discrepancy in verbal and spatial abilities in mathematics is what Dowker (2005) refers to as ‘cognitively uneven’ performance. Related to this, an extensive research literature shows that low-achieving children have deficits in their number sense (e.g., Geary et al., 2009; Landerl et al., 2004) and inabilities to use pattern recognition in enumeration (Mulligan et al., 2006). Furthermore, research also shows a strong connection between young children’s subitising skills and mathematical abilities (Desoete & Grégoire, 2006; Yun et al., 2011). Hence, rather than dismissing subset-knowers as a group of immutable low performers, the results reported in *Article 1* demonstrate how a bodily approach to subitising-based enumeration might support their abilities in exact production of small sets[14]. Accordingly, the results add to the research literature of mathematical interventions for children with immature mathematical proficiencies (e.g., Klein et al., 2008; Starkey et al., 2004).

In summary, the findings reported in the focal studies show that children’s abilities to ground mathematical thinking in embodied interaction can vary from partial knowledge and understanding to a greater degree of congruence with the targeted mathematical domains. In the next section, I will discuss further characteristics that elaborate this zone of coherence of distinct and relational number-space mappings.

[14] In *Article 1*, I also drew on the core knowledge systems of numbers (i.e., ANS, OTS; see subsection 3.1.1) to argue for underlying factors that might explain the subset-knowers’ abilities in establishing cross-modal coherence in the production of small sets.

7.2 The partial, situated, bidirectional and relational nature of number-space mappings

What characterises the partial, situated, bidirectional, distinct and relational nature of children's number-space mappings?

The two ETPs in my dissertation study were designed to foster physical experiences matching the idea of cardinality and counting-based addition, respectively. In light of this, the EC perspective holds that abstract concepts consist of several mutual supporting layers of number-space associations that are grounded in sensory-motor experiences (Barsalou, Simmons, et al., 2003; Barsalou & Wiemer-Hastings, 2005). This suggests that the programmes' selection of a particular set of number-space mappings (cf. the joint and fixed tasks in subsection 5.2.1) only reflect certain aspects of the targeting mathematical concept, and consequently indirectly hide other salient layers of the idea (Barsalou, 2003). The partial nature of number-space mappings is underlined by focal study 1, which shows that the subset-knowers were unable to transform their assessed abilities in the *Navigation task* to the *Give-N* task. This discrepancy suggests that the subset-knowers' abilities were situated and bounded to the use of *enactive metaphors* (e.g., a "frog-four"-body posture; Gallagher & Lindgren, 2015) to physically and verbally mediate the cardinal value of configurations of dots on the ground. Although the results reported in *Article 1* suggest that the subset-knowers' proficiency in equinumerosity may involve transformation between sets in different modalities (cf. subsection 3.2.3), the findings provide no support for other research suggesting that abilities in subitising-based enumeration can be an important mediator to object counting skills (e.g., Hannula et al., 2007). However, in light of the huge amount of enactive metaphors, this discrepancy points back to limitations of the *Give-N* task in terms of assessing proficiencies that go beyond the use of number words and concrete items in grounding the idea of cardinality in meaningful action (cf. the outline of knower-level theory in subsection 3.2.2). Likewise, the results reported in *Article 3* show emerging abilities in the children's full-bodily simulation of counting-based addition in the context of a 100-dotted circle, but no claims of strategy generalisation can be made (e.g., use of the min strategy in a board game). Overall, these observations underline the partial and situated nature of an embodied approach to concept formation, as any physical-based actualisation of a targeting concept can only affirm the general concept because it will always be a singular, and therefore cannot fully capture the general concept completely (see Radford, 2013).

Despite these limitations, the mutually supportive number-space associations incorporated into the embodied experiences suggest that individual weaknesses in one mode of representation might be compensated for by strengths in others. For example in focal study 1, the physical "frog-four" coupling of a squared array allow the child to infer the cardinal value *4* via mutual supporting representational domains (i.e., mental imagery of a frog posture, the verbal term "frog-four", visuo-spatial and body-spatial mapping of a four-dotted array). Support for this line of reasoning comes from the subset-knowers' assessed abilities in the *Navigation task* (cf. *Article 1*) along with a growing body of research that emphasises the multimodal nature of subitising (see the review in subsection 3.2.4). However, the picture is not clear as research also shows that children in some instances are less able to solve mathematical tasks that require inferring and connecting information from multiple representations (Ainsworth et al., 2002; Duval, 2006).

In order to further elucidate the complex multimodal interplay in children's number-space mappings, I will lean on how EC models the information flow between mind and external (beyond-the-brain) processes, that it is cyclic and bidirectional in nature and that it continuously produces feedback for the regulation of further action (Fuster, 2009). The findings in *Article 2* highlight how congruence in additive reasoning, mediated by multimodal representations and driven by complex bidirectional feedback loops, might unfold through the coordination of distinct number-space mappings into relational thinking. This is consistent with the study by Pontecorvo and Sterponi (2002), which shows that 3- and 5-year-olds' reasoning unfolds through multifaceted patterns of argumentation[15].

Another aspect of the dialectal relation between external and mental processes concerns the emergence of small and more powerful conflicting tension between newly perceived information and thinking, or what Piaget (2001) refers to as *disequilibrium*, also known as *cognitive conflicts* (Waxer & Morton, 2012).

---

[15] In section 7.3, I will along with two additional features associated with the framing question (i.e., simulated action and off-loading of thinking) provide an elaboration of the bi-directional nature of parts-whole reasoning reported in *Article 2*.

Figure 31: The emergence and solution of a cognitive conflict in the embodiment of "*monkey-three*" (ETP 1)

In *Article 1*, it was argued that small conflicting tensions in the *Navigation task* could involve renaming the physical coupling during the mapping process (e.g., "monkey, kangaroo-two", "monkey, frog-four" and "frog, cock-a-doodle-doo, monkey-three") or tagging extensions to include more dots (e.g., saying "kangaroo, frog-four" while expanding a bipedal to a quadruple coupling). A more notable example discussed in *Article 1* regarding how physical interaction might shape mathematical thinking involved C2-knower Liv's[16] solution of a cognitive conflict when articulating "frog-four" in the physical tagging of three dots (see also Figure 31). One hypothesis put forward was that the intermodal mismatch provided feedback to adjust the speech to match the action (i.e., "monkey-three"), thereby establishing congruence between the verbal and body-spatial domain[17]. This line of reasoning is supported by evidence that shows that the dialectical nature between cognition and newly perceived information might produce and solve conflicting tensions (Herawaty & Widada, 2017; Prusak & Hershkowitz, 2019). However, in contrast to other studies that emphasise reflection as a main way to promote the emergence and resolution of cognitive dissonance (Young & Shtulman, 2020), the results in *Article 1* show how perception from children's own speech and sensory-motor engagement may support such epistemic processes (cf. Harmon-Jones et al., 2009; Schwarz & Prusak, 2016).

To summarise, the observations above highlight two additional features of the framing question. First, they demonstrate that an embodied approach to mathematical thinking is limited in terms of its situated nature allowing the child only to discern certain patterns and structures (generalities) associated with the targeting domains (cf. Mulligan & Mitchelmore, 2013; Radford, 2010). Second, they show how actions and interactions embody projections of the mind and vice versa (Lakoff & Johnson, 1999; Sriraman & Wu, 2020), thereby underlining the epistemic role the bidirectional relation of number-space mappings possesses in embodied situating of thinking.

[16] The names used in this discussion are anonymised and the respective articles provide detailed information of the children mentioned.
[17] See *Article 1* for discussion of alternative hypotheses.

7.3 Number-space mappings in simulated action and off-loading of thinking

What characterises children's re-enactment and off-loading of number-space associations into gestures, body parts, bodily movement, spatial affordances and use of tools?

Two fundamental principles of the EC framework might further illuminate characteristic features of the situating of number-space associations in bodily action. The first assumption holds that humans off-load cognitive work onto the environment, the second posits that mental simulations and visualisations, or what Wilson (2002) refers to as *off-line cognition*, is body based (Barsalou, 2003, 2008). Support for the first of these principles comes from focal study 3 (see *Article 3*), where I identified three categories of how 4- and 5-year-olds' might off-load their additive thinking onto tools (i.e., dice), gestures (i.e., touch counting) and full-body-spatial interaction in the 100-dotted circle. In particular, it was shown that some relied on the bidirectional mapping between mental representation of the value of the handheld dice and full-body action to support coherence with the min strategy, while others opted for touch counting or visual perception of one unit at a time to guide the body based addition forward.

Refinement to both of these principles of EC comes from focal study 2 (see *Article 2*), where the results from the *Jumping task* show that the young children, across solution skills, off-loaded their additive reasoning on spatial structures (i.e., linear or canonical patterns), gestures, pointing-trajectories and tools. These findings are consistent with a growing body of educational research highlighting that additive thinking is not an abstract process, but rather largely an issue of reasoning about entities and spatial structures located in space (e.g., Cheng & Mix, 2014; Kullberg et al., 2020). In addition, the results reported in *Article 2* suggest that most children (i.e., 8 out of 10) were able to transform and recreate the sensory-motor experiences from ETP 1 in several stages of task exploration, for example by mimicking the full-bodily experiences in structured finger patterns (i.e., all-at-once gestures, point-counting), in (de)contextualising and modelling to relieve the cognitive work. These findings are consistent with studies showing that gestures can ease the cognitive load when hand movement simulates the targeting area in a meaningful manner (Cook et al., 2012), and that gestures might ground mathematical thought in action (Beilock & Goldin-Meadow, 2010) and bring implicit mathematical knowledge to learning (Broaders et al., 2007). In order to underline that reason is not disembodied "but arises from the nature of our brains, bodies, and bodily experience"

(Lakoff & Johnson, 1999, p. 4), *Article 2* provides several illustrative examples, some of which I will outline and discuss below.

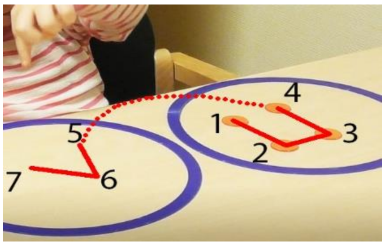

Figure 32: Canonical patterned point counting (*4+y=7*) – from focal study 2

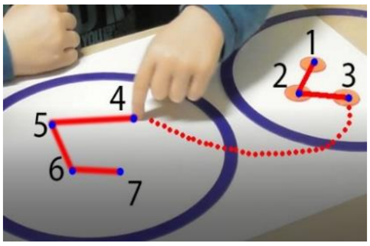

Figure 33: Extension from six to seven touches *(3+y=7)* – from focal study 2

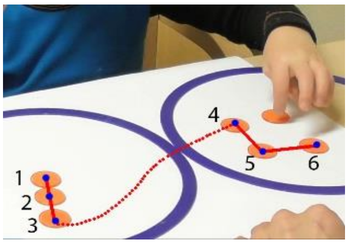

Figure 34: "I must remove this" (*4+y=6*) – from focal study 2

First, on the *4+y=7* task[18], without stating any guesses, Elli started directly on patterned point counting, first over the squared array verbally expressed as "One, two, three, four" and then over the empty circle "five, six, seven" to produce a triangular shape, followed by the quick response "monkey-three" (see Figure 32). Second, based on the guesses "monkey-three" on the *3+y=7* task, Leo's rationale for answering "frog-four" was grounded in an extension of the triangular patterned touch counting trajectory in the empty circle to make a quadrangle when he realised that *six* was *one* less than the requested sum of *seven* (Figure 33).

The materialisation of this class of parts-whole reasoning suggests a close bidirectional interplay between mathematical thinking and simulated action in terms of continuous comparison of the current articulated number word, the requested sum, spatial mental models of body postures and the generated movement trajectory in the empty circle for mental off-loading and driving the reasoning forward. This suggestion finds support in research showing that reasoning and problem solving rely on sensorimotor simulations (e.g., Alibali et al., 2014). A rival or possibly a complementary explanation is that inaudible counting may have preceded mental retrieval of the spatial structures of the body postures and the articulated gestural behaviour[19]. Supporting evidence for this proposal comes from Thevenot et al.'s (2016) study, which shows that 10-year-olds can use fast internal counting when solving single-digit addition problems. Data on children below 5 years of age, however, are sparse. On the other hand,

[18] On the *4+y=7* task, the interviewer asks: "If you jump a frog-four here (pointing at the circle with a squared array of items) and then you jump something here (pointing at the empty circle) before you jump out, what did you jump here (pointing at the empty circle) if you jumped seven all together?"
[19] See *Article 2* for discussion of alternative hypotheses.

deviant examples[20] show that simulated action through an extensive off-loading of the reasoning for those opting patterned modelling also could hinder the fluency in problem solution. This suggests that the degree of mental off-loading for the 'patterned counters' spans a continuum, ranging from extensive to limited use of tools and gestures for scaffolding the additive reasoning.

By contrast, two CP-knowers show via linear-based reasoning and problems in contextualising independence of the patterned embodied experiences from ETP 1 in solving the *Jumping task* (cf. Figure 34). A partial explanation is that the children's linear way of thinking and representing numbers was too conceptually robust and functional for the emergence of an alternative way of modelling. Support for this line of reasoning comes from research showing that children are reluctant to replace internalised strategies (Gray et al., 2000; Ostad, 1998). A complementary hypothesis addressed in *Article 2* concerns that the choice of animal behaviour as a theme in ETP 1 reflected the children's prior knowledge in a biased way. Consequently, since children are best at learning the names of objects and actions they are interested in (Yu, 2014), the question of whether the selection of more familiar themes might have provided more meaningful number-space associations, especially for those opting for linear modelling, is yet to be explored. Despite these diversities, the findings align with research showing a strong link between visual-spatial abilities and mathematical reasoning (e.g., Hawes et al., 2017; Hegarty & Kozhevnikov, 1999), and it concurs with a growing body of educational research suggesting that knowledge of numbers structured as parts and wholes is foundational for arithmetic reasoning abilities (e.g., Hunting, 2003; see the review in section 3.3). Finally, the findings add to the literature on how gesturing might lighten the cognitive load in explaining math (e.g., Goldin-Meadow et al., 2001) and help young students in solving missing-addend problems (e.g., Goldin-Meadow et al., 2009; Novack et al., 2014).

To summarise, the following three characteristics add to our understanding of young children's grounding of mathematical thinking. Firstly, the findings in this dissertation study show that children's mathematical thinking can be off-loaded onto movement trajectories, body parts, gestures, spatial structured affordances and available tools in various ways, some more efficient than others. Secondly, the results show that the children's argumentation are driven by complex feedback loops in which distinct number-space mappings change the condition for further

[20] For example, on the $4+y=6$ task, Noah used manipulatives and all-at-once gestures for modelling and a canonical patterned counting-all-strategy to explore each of the guesses *frog-four, cock-a-doodle-doo-one* and *kangaroo-two* (see *Article 2*).

action, thereby making the foundation for relational mathematical thinking. Finally, the findings show that simulation of full-body experiences possesses the potential to influence both the content, the order and the coherence of the mathematical arguments made.

7.4 Characteristics of efficiency in number-space mappings

What characterises efficiency in children's bodily grounding of mathematical thinking?

Cognition is time pressured and *Cognition is for action* are two theoretical principles from Wilson's (2002) classification of EC that I will use to discuss characteristic features of effectiveness in the bodily situating of mathematical cognition. I will argue that the basic criteria of efficiency was that it functioned under the pressure of situational and contextual factors such as real-time demands emerging from rapid bodily appropriation of spatial affordances, and that the cognitive work served rather than hindered the physical interaction as in the case of ineffective task solution. Additional features of inefficiency include time for observation and extensive offloading of the mathematical thinking.

The results reported in *Article 3* show that ineffective physical modelling of the min strategy was associated with rigid movement patterns (i.e., slow gait) and extensive use of gestures and/or tools in order to keep track of counted units, while efficiency was related to motor fluency, bodily coordination and mental representations of the addends. The link between strategy efficiency and mental retrieval and representations finds support in a huge body of research (e.g., Ostad, 1997; Threlfall, 2009). In addition, the findings reported in *Article 3* show that attributes of efficiency could involve expressive body postures to represent the largest addend and the sum, while long jumps and rhythmic movement patterns could connect distinct number-space mappings into relational additive thinking.

Figure 35: Motor fluency – from focal study 3

Figure 36: Bodily rotation in tagging the sum – from focal study 3

Figure 37: Time pressure from jumping – from focal study 1

Figure 38: Bodily adaption – from ETP 1

In *Article 3*, I used Anna's physical situating of the min strategy on the *6+3* task to exemplify this, where the largest addend and the ordinal structure (see Figure 35) were materialised in a rhythmic movement pattern "six, seven, eight" followed by a forceful bodily rotation in mapping the sum "nine" (see initial phase in Figure 36). Modulated speech synchronised with flow in locomotion suggests a high degree of engagement, which as demonstrated by Anna's task behaviour can entail gross-motor rhythm and expressive movement patterns (e.g., force, tempo, fluency, twist). This provides support for claiming that the cognitive work associated with this class of additive modelling fostered rather than inhibited the physical fluency, thereby providing a subjective layer of meaning to the children's mathematical thinking. This is consistent with EC's view that conceptual knowledge is rooted in many simulations that continuously add subjective and objective (i.e., generalities) layers of meaning that come from the synthesis and coordination of movement and actions that project central aspects of the targeting concept (Lakoff & Núñez, 2000; Radford, 2010, 2013). In line with Radford (2015), this suggests that embodied rhythm might be an integrated part of mathematical thinking. In contrast, unitary visual or tactile interaction with the handheld dice to guide each step in the ordinal part of the min strategy suggests that the children used their cognitive and motor resources on the complex synchronisation of eye, dice, feet and spatial layout. This is consistent with research showing that the use of unitary counting strategies (e.g., Gray et al., 2000) and an increase in working memory load (e.g., Wang & Shah, 2014) usually decreases efficiency in math performance. In *Article 3*, I also argued that a monotonous speech in combination with limited opportunities to utilise existing motor skills, materialised in rigid movement patterns, may increase the child's threshold for perceiving personal relevance and meaning of the action. In general, these lines of reasoning concur with studies that warn that cognitive demands and unnecessary cognitive load may be a risk for embodied learning (Ruiter et al., 2015; Skulmowski et al., 2016), suggesting that less embodied cues may in some epistemic instances may be preferable to an overload of multi-sensory input.

The results reported in *Article 1* show that children who jumped from one physical coupling to the next were able under the pressure of time (see Figure 37) to coordinate speech with rapid and flexible adaption of body postures to available configurations of dots (see Figure 38). In contrast, slow bodily situating of number-space associations was reflected in walking around in the circle searching for arrangements of dots that fit rigid ideas about the configuration of potential arrays. *Article 1* provides several examples that illustrate temporal differences in grounding the idea of cardinality in body-spatial interaction in the 50-dotted circle. For example, C3-knower Max (*Walker*) used 35 seconds in producing the series "*monkey-three, frog-four and kangaroo-two*" (notated as *3+4+2*), while C2-knower Liv and C3-knower Amy (both *Jumpers*) used 18 and 12 seconds in the physical mapping of six (*1+2+1+3+4+2*) and five (*1+2+3+4+4*) sets, respectively[21]. The mathematical thinking for jumping children was associated with flexible motor adaption of the spatially distributed affordances, which in turn underlines the ultimate contribution that cognitive activity can play in situation-appropriate behaviour (Johnson-Glenberg et al., 2014). In contrast, the *Walker's* problems in pattern recognition seemed to prevent them from benefiting from existing motor skills. Related to EC's claim that *Cognition is for action*, these diametrical observations in spatial-temporal appropriation highlight challenges in designing bodily tasks that cultivate rather than prevent the use of motor skills in ecological meaningful ways to achieve pedagogical goals.

The Caviola et al. (2017) survey study shows that time pressure generally works as a stressor causing anxiety and suboptimal performance, and consequently has a negative influence on strategy efficiency in arithmetical tasks. This line of evidence stands in contrast to my findings (focal study 1 and 3), which show that time pressure was self-regulated (e.g., *Walkers* vs. *Jumpers* in focal study 1), it was connected to efficiency in math performance and partly to personal relevance through expressive movement patterns and use of motor skills. Based on these arguments, my study provides case-based evidence that the claims of EC that cognition is for action and exposed to time pressure from real-time action can shed light on characteristics of children's grounding of mathematical thinking regarding efficiency and perception of personal meaning (Wilson, 2002).

7.5 Synthesis of characteristics of children's grounding of mathematical thinking

The first part of this section provides a synthesis of the different aspects of the framing question discussed above (sections 7.1-7.4), and it concludes with a proposal for progression paths that can inform KTs about profiles of young children's abilities to ground mathematical thinking in embodied interaction.

---

[21] The 'Jumpers' and the 'Walkers' used respectively 4.1 seconds and 8.9 seconds in average for each tagged set of 1 to 4 dots (cf. Article 1).

The findings reported in the three articles included in my dissertation study show that young children's grounding of mathematical thinking is a complex issue elaborated in the details of embodied action and concurrent use of multimodal resources (cf. Goodwin, 2000; Hutchins, 2006). Promoting and inhibiting characteristics associated with congruency and fluency in distinct and relational number-space mappings in what is described as a 'zone of partial knowledge and understanding' (cf. Dowker, 2005) span from erroneous, via ineffective to effective ways of modelling mathematical thinking in embodied interaction. Incorrect task behaviour, or what Dowker (2005) refers to as 'cognitively uneven' performance, include action-speech mismatches and inabilities in connecting distinct number-space mappings to verbalised relational thinking. Supporting aspects include flow in movement, expressive body movement and postures, flexible bodily adaptation to spatial structured affordances, modulated speech, mental arithmetic and that the thinking worked under time pressure from real action. Signatures of ineffective task behaviour include monotonous speech, inflexible bodily adaption to spatial affordances, delayed body movements and impassive body postures, and that the thinking did not work under time pressure of 'normal' physical behaviour (e.g., gait), extensive off-loading of thinking, and rigid coordination of the multimodal correspondence between ordinal and cardinal properties of numbers. However, it is important to note that behavioural patterns labelled ineffective do not necessarily imply incongruence with the targeting mathematical domains. Rather, the range of constellations of number-space mappings can be viewed as an epistemological strength, as it allows the child to use preferred physical and mathematical abilities. Although this suggests that there does not exist a unique path to ground mathematical thinking in embodied action, the identified categories of task behaviour in the focal studies may point on the conjectures of progression paths toward an efficient, congruent and meaningful grounding of mathematical thinking. In focal study 1, stages in such a progression path may include non-verbal body-spatial production of small sets as a foundation for congruent cross-modal mapping of numerosity in a delayed mode (i.e., walking) followed by an efficient, flexible and rapid bodily adaption to the structured affordances (i.e., jumping) in the grounding of the idea of cardinality in time-pressured real action (cf. *Article 1*). In focal study 3, a progression path may initially involve assembling distinct number-space mappings into relational thinking according to the rules of the min strategy, where touch-counting and visual retrieval of numerical information from the handheld dice to guide the ordinal structure of the min strategy can function as intermediaries for mental based strategies that may include the signatures of effective task behaviour expressed above (e.g., expressive body movement; cf. *Article 3*). To summarise: The characteristics associated with children's embodied modelling of mathematical thinking provide a theoretical contribution to the EC framework (e.g., the principles of EC stated by Wilson, 2002), and they add to the debate about how young children's structured-based experiences of cardinal and ordinal properties of numbers can serve as basis for powerful arithmetic skills (e.g., Björklund, Ekdahl, et al., 2021). Below, I will make further reflections on the contributions from my dissertation study.

7.6 Contributions from the dissertation study

Today, there is agreement on the importance of early cultivation of mathematical knowledge for later academic achievement (e.g., Aunio & Niemivirta, 2010; Parsons & Bynner, 2005). To contribute to the ongoing debate about what high-quality mathematical interventions entail, this dissertation study has investigated young children's bodily grounding of mathematical thinking in embodied designs situated outdoors in two Norwegian ECEC institutions. Previously, I used Skulmowski and Rey's (2018) taxonomy on embodiment in education to position the DBR part of my dissertation study in the quadrant of their 2x2 grid model that involved high degree of bodily engagement and high degree of task integration (section 4.3.2). It is important to note that the authors base their notion of meaningful embodied learning on whether the bodily representation is deeply integrated into the learning task or whether it is an incidental aspect (Skulmowski & Rey, 2018). Based on this, I hold that the suggested progression paths (which are informed by the three focal studies) involving full-body action and objective and subjective dimensions of meaning (cf. DP 1 and 2, respectively) provide some quality characteristics that add to the literature about embodied interventions in non-digital contexts and principles guiding such research. In particular, the results on the integration of composite movement patterns and expressive body movements in mathematical thinking are of interest in terms of fostering the phenomenological dimension of meaning in young children's mathematical engagement. Related to this, Whitacre et al. (2009) used the metaphor *student-as-artist* to distinguish expression from representation in mathematical embodied activity, thus emphasising the creative and personal dimension in mathematical learning. Applied to this dissertation study, the results demonstrate how the *child-as-artist* metaphor draws attention to expressive layers of modelling mathematical thinking (i.e., physical tempo, rhythm, force and flow, body-postures, modulated speech and drama of animal behaviour). The *child-as-artist* metaphor also applies on the ETPs, which incorporated music and dance into the body-based learning activities (e.g., 'Animal farm' and 'Rhythmic tagging'; see subsection 5.2.1). This suggests that my dissertation study may contribute to a debate about the inclusion of expressive body movements in powerful ways of modelling mathematical thinking (cf. Whitacre et al., 2009).

In the literature review of *Embodied Numerical Cognition* (subsection 3.1.2), I identified inquiries focusing on part-body- or upper body movement and full-body movement as two dominant lines of research that aimed to capture the role of the body in number-space mappings. The survey involving full-body movement also showed a bias towards interventions modelling the linear structure of the MNL in computer assisted one-to-one settings indoors (Tran et al., 2017). Based on this, my research shows that the notion of number-space mappings is more diverse and multifaceted as it reveals that children can ground mathematical thinking in different multimodal constellations in non-digital outdoor settings while moving and acting in three dimensions (*Article 1* and *3*). It also shows how full-body and upper-body movement can be connected via simulated action in parts-whole reasoning (*Article 2*). Hence, rather than treating movement as a mere motivational element in achieving pedagogical goals, my research highlights the interacting body as an integrated part of mathematical thinking, thereby adding to our understanding of the notion number-space mappings (Cipora et al., 2018; Patro et al., 2014) and to the literature focusing on the role of the moving body in mathematical learning (e.g., Beck et al., 2016; Fischer et al., 2011; Moeller et al., 2012). Notably, Wilson and Golonka (2013) view studies (such as my dissertation study) that focus on the concurrence of cognitive processes and physical resources as more valuable for embodied research than investigations testing hypotheses of how bodily influences can prime cognitive performance. My findings also add to the field of knowledge concerning young children's structured-based learning of elementary arithmetic within the number range 1 - 12 (e.g., Cheng & Mix, 2014; Jansen et al., 2014; Schöner & Benz, 2017). Under this line of research, Björklund, Marton, et al. (2021) identified modes of number representations, ordinality, cardinality and part-whole relation as critical aspects necessary to discern for children to develop powerful arithmetic skills. However, in contrast to studies that emphasise finger gestures as modal representation (e.g., Björklund et al., 2018; Kullberg et al., 2020), my dissertation study provides knowledge of how full-body-spatial interaction might support young children's thinking about cardinality, ordinality and parts-whole relations of numbers.

In the research domain of early assessment in mathematics, there is consensus that young children's mathematical abilities are not easily assessed as they are able to solve mathematical problems that they cannot talk about. Accordingly, mathematical tasks that have a strong literacy base might disadvantage children with other modal preferences (e.g., visual, auditory and kinaesthetic; Clausen-May, 2005). From an EC perspective, a main issue for KTs in regard to formative assessment is therefore to capture signatures of how whole-body movements are related to targeting mathematical domains. This seems particularly important for children considered low performers according to traditionally assessment tasks in mathematics (Houssart, 2013). The theoretical foundation for the assessment tools used in this dissertation is anchored in the EC framework and the DPs (section 4.3), and I will argue that this battery of measures adds to the literature of early assessment (cf. Purpura & Lonigan, 2015). In particular, the modified *Give-N* task in focal study 1 (subsection 5.3.1) contributes to the knower-level theory (Lee & Sarnecka, 2010; Sarnecka & Carey, 2006) as it highlights a multimodal approach for measuring children's idea of cardinality in outdoor settings. The EC approach is further highlighted by the *Navigation task* in focal study 1 (subsection 5.3.1), which in contrast to the standardised *Give-N* task builds on children's imitation abilities, autonomy (i.e., free choice of sets) and skills in recognition and adapting body-postures to spatial structured affordances (instead of picking out bricks from a pile). In focal study 2, I developed a procedure for measuring children's abilities in additive reasoning that might include simulations of first-hand experiences with numbers (cf. *Jumping task*), while the *Min task* in focal study 3 measured children's abilities in whole-body modelling of counting based addition (subsection 5.3.1). Accordingly, these procedures might be used to assess children's guided focus on mathematics (cf. Hannula et al., 2005). Hence, informed by the suggested progression paths (section 7.5), the formative character of these measures reflected in a close connection to the ETPs, supports the blending of children's play- and movement-based learning with KTs observation and guidance. Such efforts of incorporating guidance and learning into (KT structured) play, routines and everyday activities are what Tate et al. (2005) refers to as *embedded teaching*. Based on these reflections, I hold that my research contributes to the debate about embedded teaching approaches in early childhood education (Pramling et al., 2019). However, as argued by Clements and Sarama (2011), there are huge differences in how children are provided with opportunities to pay explicit attention to mathematical concepts in their informal experiences. Therefore, well-educated KTs in an EC approach to early learning is required, which in turn rests on how professional development courses, kindergarten teaching and policy documents direct attention to the key findings of my dissertation study. Moreover, teachers in primary school can also benefit from insight from this study. However, a smooth transition from kindergarten to primary education requires continuation in pedagogy and curriculum (OECD, 2017), which in this case should be reflected in the fostering of bodily experiences in game based contexts for modelling mathematical thinking. This aligns with the Norwegian

Knowledge Promotion Reform for compulsory education, and its emphasis on practice and play based learning activities for the youngest children (Norwegian Ministry of Education and Research, 2020). According to this curriculum, the pupil is expected after year 2 to be able to represent numbers in different ways and switch between different representations (cf. *Article 1*), follow rules and step-by-step instructions during play and games and experiment with counting and choosing different starting points (cf. *Article 3*). Likewise, it is expected that the pupil after year 3 can explore equilibrium and balance in practical situations, and represent this in various ways, and after year 4 explore relationships between addition and subtraction and use this in mental arithmetic and problem solving (cf. *Article 2*). Together, these observations provide many reasons to use key findings from my research to cultivate the link between play, bodily experiences and mathematical thinking in primary school. However, I must note that the suggested learning path is a working hypothesis, which along with the test procedures and activities included in this research must undergo rigorous scientific examination to assess its usefulness across contextual conditions and groups of children in different ages (cf. Wikfeldt, 2016). This is consistent with the conclusion of the review study on SFON[22] of Rathé et al. (2016), that more attention is needed to the design of guided learning activities and valid ways of assessing children's mathematical proficiencies in play and daily activities.

As argued by Björklund et al. (2020), the goal of developing deep knowledge of the emergence of young children's mathematical knowledge places high demands on research methods. Based on this, I contend that the methods used in this dissertation study are relevant for educational research on young children in outdoor settings as detailed descriptions of interventional activities (cf. fixed and joint task) and rigorous descriptions of the procedures used for data collection support replication of the study (Yin, 2009). Moreover, the use of micro-analyses of video footage allowed me to develop a coding structure that accounted for the relation between simultaneous and connected number-space mappings (cf. Ekdahl et al., 2016), and the use of case-methodology (i.e., pattern matching, cross-case synthesis and explanation building through multi-case analyses; Yin, 2009) allowed me to identify and examine categories of grounding mathematical thinking in talk and embodied interaction. In this way, the methodological approach accounted for the complexity in children's bodily grounding of thinking, but it also enabled me to consider shared features and patterns across individuals for

[22] The notion of *Spontaneous Focusing On Numerosity* (SFON) refers to children's spontaneous unguided (linguistic) capacity to pay attention on exact numerosities in their environment (Hannula, 2005).

analytical generalisation of my findings using principles from the EC framework. Based on these reflections, I hold that my dissertation provides methodological insights into how a qualitatively research approach can develop our understanding of children's grounding of mathematical thinking in embodied action (see additional arguments in section 5.5).

7.7 Limitations, further research and implications for practice

Although this dissertation study is only a start towards unravelling the relations between different forms of embodied experiences in the situating of number-space associations, it has raised some practical and theoretical questions that may inspire future research. For instance, the findings reported in *Article 1* and *2* suggest that embodied actualisations of mathematical knowledge embedded in the activities in the ETPs might support the transformation of situated knowledge across contexts and content domain (cf. Fugate et al., 2018). The results also show how the DPs (see section 4.3) can be the starting point for developing a series of activities that together foster young children's bodily grounding of specific mathematical targeting areas. However, the list of activities is not a prerequisite for working with the mathematical targeting domains addressed in the two programmes, as the findings underscore that any embodied grounding of mathematical thinking (a particular) only mediates partial aspects of the targeting mathematical concepts (Barsalou, 2008; Lakoff & Núñez, 2000; Radford, 2013). Therefore, KTs need to carefully consider what the children need in terms of activities, guidance, tools and personal, social and conceptual layers of meaning when situating mathematical thinking in full-body interaction. This in turn points to the dynamic and temporary nature of the DPs, suggesting that future research should test, evaluate and refine the foundations for the design of embodied learning environments.

An interesting finding in my research is the inclusion of expressive body movements (e.g., rhythm, force, tempo, fluency) in the children's grounding of mathematical thinking. This result about *children-as-artists* in embodied mathematical learning (cf. Whitacre et al., 2009) is important for further discussions from both a pedagogical and embodied theoretical point of view. Although the review of educational body-based research in mathematics (cf. subsection 3.1.2) shows no sign of including expressiveness as part of the investigated phenomena, it shows a bias towards interventions cultivating linear (side-ways) movements in modelling the MNL (e.g., Dackermann et al., 2016; Fischer et al., 2011; Link et al., 2013). Hence, the question if this boundary set for moving in space also entails limitations for the inclusion of the *child-*

as-artist in early embodied learning of mathematics, is open for further research. A challenge for the EC framework is therefore to incorporate subjective dimensions into the modelling of mathematical thinking in terms of theorising how expression and representation might merge and support children's epistemic processes. A practical implication (cf. the *child-as-artist* metaphor) is that KTs can direct the child's attention to the mathematical content in play and everyday activities that have the potential to involve bodily rhythm and creativity. Related to this, research shows a strong correlation between young children's motor life skills, play skills and mathematical abilities (Reikerås, 2020; Reikerås et al., 2017), suggesting that future research should investigate how interventions can support low achievers to develop competences across these areas. Moreover, Reikerås and Salomonsen (2019) found that only 25% of the children who were assessed to be among the weakest 10% at toddler age, were in the group with the weakest mathematical abilities in preschool age. This suggests that the level of mathematical abilities for young children is unstable over time and that toddler age might be too early to predict later difficulties. Hence, the authors call for more research to investigate why the high proportion of toddlers performing at an acceptable level turn out as pre-schoolers with weak mathematical abilities. This points to the complexity of early assessment, but possibly also to weaknesses in the tools and approaches used for measurement. Therefore, I recommend that future research should develop procedures for how observation and measurement of children's number-space mappings might be integrated in joint learning activities in outdoor embodied designs, and that the gross-motor imitation part of these activities can serve as a basis for sense-making independent of the child's competence (cf. DP 4). In this regard, I think that the large-scaled multi-dotted circle is a promising starting point, as my dissertation study demonstrates its applicability in the incorporation of children's culture of play in the embodied learning activities and in observation and assessment of specific mathematical targeting domains. This is consistent with Malinverni et al.'s (2014) systematic review of design studies on learning of abstract concepts through whole-body interaction and the author's suggestion that the development of adequate tools for assessment should be combined with analysis of the design choices, behaviour and cognitive processes.

In the Caviola et al. (2017) review study of how time pressure manipulation can interfere with strategy selection in arithmetical tasks, the authors conclude that although the survey shows that time pressure has a great influence on both strategic and emotional aspects of task performance, very few studies consider executive functions and cognitive processes. In this regard, I think my findings connected to effectiveness in children's grounding of mathematical thinking can inspire further investigations into the relationship between mental arithmetic and embodied tasks that enable self-regulation of time pressure (cf. the *Jumping task* and the *Min task* in subsection 5.3.1). There is also need for more DBR that builds on the blending of structural (e.g., Björklund et al., 2018; Kullberg et al., 2020) and embodied (e.g., Fischer et al., 2011; McCluskey et al., 2018) approaches to early learning of numbers and arithmetic.

However, as mentioned earlier, the wrapping of mathematical ideas in embodied action should always consider that some children might experience shortcomings and failures in their attempts to make sense of the complex network of cross-modal number-space mappings. Hence, embodied learning is not necessarily desirable for all children (Tran et al., 2017). In addition, strong rules and guidance can destroy the children's motivation for outdoor play. Therefore, I recommend that any embodied intervention consider the fragile balance between empowering the children's culture of play, motor proficiencies, joy, feelings and freedom of being outdoors against degrading these core aspects that define the children as subjects with their own personal values, needs and goals. I think a main goal in this regard is to promote children's awareness and abilities in using their body as a flexible means of meaningful mathematical thinking, communication and play with peers, for a better understanding of the world around them and for personal and social development. Based on these reflections, I think that the results of my dissertation study give reason to rethink the role that the moving body can play in outdoor learning of mathematics in Norwegian ECEC institutions.

To summarise, although the dissertation study shows some promising results, much is still unknown regarding the direction of influence between motor activation, environment and mathematical thinking (cf. Shaki & Fischer, 2014). However, in light of current theories about the role of the EC perspective in supporting the emergence and retrieval of number-space associations, I hope that my dissertation study will encourage further research on how full-body movement can contribute to and explain young children's mathematical thinking in outdoor contexts.

7.8 Summary

Based on an EC perspective, the main objectives of my dissertation were to deepen the understanding of the embodied grounding of three mathematical domains considered important to foster in early years, and to develop knowledge of how outdoor embodied designs can facilitate meaningful number-space associations targeting these areas.

Although my research project has generated more hypotheses and new questions than exact answers, the findings have in several ways added to the literature of educational research that focuses on the role of the moving and acting body in young children's mathematical thinking. Firstly, it demonstrates deviant and recurring patterns of kindergarteners' structure-based bodily situating of cardinality and addition, and in the transformation of physical experiences of numerosities across content domain via gestures and manipulation with tools in parts-whole reasoning. In this way, my study underlines previous findings suggesting that structured-based experiences with numbers are foundational to young children's development of arithmetic skills (e.g., Björklund, Marton, et al., 2021; Venkat et al., 2019), and it underscores the dialectical nature of outdoor physical experiences and simulated action in mathematical thinking (Wilson, 2002). Secondly, it shows how distinct and connected number-space mappings might be rooted in young children's everyday experiences (e.g., walk, jump, body-gestures, manipulation with tools) in a manner that cohere with the logic and rules of ordinality, cardinality and arithmetic. Thirdly, it shows how children's upper- and full-body spatial interaction, use, and manipulation with tools might extend and load mathematical ideas and reasoning onto spatial extensions and locations. Fourthly, it demonstrates the epistemic power of the dynamic and bidirectional relation of number-space mappings, allowing the child to experience the targeting concepts with multiple cues, representations and layers of meaning and for fostering the emergence and resolution of conflicting tensions between thinking and experiences. Fifthly, it shows that the basic criteria for effectivity in the bodily situating of mathematical thinking was that it functioned under the pressure of real-time interaction with the environment. Next, and most interestingly I think is that the study reveals how the body can convey mathematical content in combination with complex and expressive movement patterns, which in turn allows the child to perceive the cognitive work as a means to provide personal meaning and relevance. Finally, based on the partial and situated nature of embodied actualisations of mathematical thinking, the findings also demonstrate that the young children's bodily grounding of mathematical thinking provide limitations in the robustness and generalisability of the learned content.

I positioned the DBR part of my dissertation study in the category of educational embodied studies involving a high level of task integration and bodily engagement (Skulmowski & Rey, 2018). In this regard, I think that the main contribution is that my results show how the design principles can guide the (re-)design of activities that enable the children to experience abstract mathematical targeting domains with mutually coexisting layers of meaning including elements from children's culture of play and existing motor skills. In particular, the whole-body approach was fruitful as it underlines that learning for young children can be associated with finding new ways of moving and interacting in space. In addition, my dissertation study shows that children with diverse mathematical proficiencies might work with the same embodied activities outdoors, and that the cultivation of these first-hand experiences (cf. ETP 1) can range from basic quantification (cf. the *Navigation task* in focal study 1) to more complex arithmetic reasoning (cf. the *Jumping task* in focal study 2). In this regard, I think that the strongest argument for this dissertation study concerning children labelled as low, medium or high-performers, is that the results show the conjectures of the integration of learning and assessment in physical activities outdoors across competencies (cf. the suggested progression paths), thereby contributing to counteract a trend towards indoor testing and scholastic teaching in ECEC institutions (cf. section 2.1). Although much remains unknown about how the physical body and its intrinsic dynamics can contribute to and explain young children's mapping of mathematical ideas in space, the EC approach provides a suitable framework for the qualitative in-depth study of young children's mathematical thinking in dynamic settings outdoors in ECEC institutions. In conclusion, the dissertation study has deepened our understanding of young children's grounding of mathematical thinking in embodied action.

References

Abdullah, N., Zakaria, E., & Halim, L. (2012). The Effect of a Thinking Strategy Approach through Visual Representation on Achievement and Conceptual Understanding in Solving Mathematical Word Problems. *Asian Social Science, 8*(16), 30-37. https://doi.org/10.5539/ass.v8n16p30

Abrahamson, D. (2013). *Toward a taxonomy of design genres: fostering mathematical insight via perception-based and action-based experiences* Proceedings of the 12th International Conference on Interaction Design and Children, New York, USA. https://doi.org/10.1145/2485760.2485761

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. *Cognitive Research: Principles and Implications, 1*(1), 33. https://doi.org/10.1186/s41235-016-0034-3

Abrahamson, D., & Lindgren, R. (2014). Embodiment and embodied design. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (Vol. 2, pp. 358-376). https://doi.org/10.1017/CBO9781139519526.022

Abrahamson, D., Trninic, D., Gutiérrez, J. F., Huth, J., & Lee, R. G. (2011). Hooks and shifts: a dialectical study of mediated discovery. *Technology, Knowledge and Learning, 16*(1), 55-85. https://doi.org/10.1007/s10758-011-9177-y

Adenzato, M., & Garbarini, F. (2012). Embodied Cognition. In N. M. Seel (Ed.), *Encyclopedia of the Sciences of Learning* (pp. 1114-1116). Springer US. https://doi.org/10.1007/978-1-4419-1428-6_900

Agostini, E., & Francesconi, D. (2020). Introduction to the special issue "embodied cognition and education". *Phenomenology and the Cognitive Sciences*. https://doi.org/10.1007/s11097-020-09714-x

Ainsworth, S., Bibby, P., & Wood, D. (2002). Examining the effects of different multiple representational systems in learning primary mathematics. *The Journal of the Learning Sciences, 11*(1), 25-61. https://doi.org/10.1207/S15327809JLS1101_2

Ali, A. M., & Yusof, H. (2011). Quality in qualitative studies: The case of validity, reliability and generalizability. *Issues in Social and Environmental Accounting, 5*(1/2), 25-64. https://doi.org/10.22164/isea.v5i1.59

Alibali, M. W., Boncoddo, R., & Hostetter, A. B. (2014). Gesture in reasoning: An embodied perspective. In L. Shapiro (Ed.), *The Routledge handbook of embodied cognition* (pp. 150-159). Routledge. https://doi.org/10.4324/9781315775845

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247-286. https://doi.org/10.1080/10508406.2011.611446

Altheide, D. L., & Johnson, J. M. (1994). Criteria for assessing interpretive validity in qualitative research. In *Handbook of qualitative research.* (pp. 485-499). Sage Publications, Inc.

Anderson, M. L. (2003). Embodied cognition: A field guide. *Artificial intelligence, 149*(1), 91-130. https://doi.org/10.1016/S0004-3702(03)00054-7

Ando, A., Salatino, A., Giromini, L., Ricci, R., Pignolo, C., Cristofanelli, S., Ferro, L., Viglione, D. J., & Zennaro, A. (2015). Embodied simulation and ambiguous stimuli: The role of the mirror neuron system. *Brain Research, 1629*, 135-142. https://doi.org/10.1016/j.brainres.2015.10.025

Andrews, P., & Sayers, J. (2015). Identifying Opportunities for Grade One Children to Acquire Foundational Number Sense: Developing a Framework for Cross Cultural Classroom Analyses. *Early Childhood Education Journal, 43*(4), 257-267. https://doi.org/10.1007/s10643-014-0653-6

Anobile, G., Arrighi, R., & Burr, D. C. (2019). Simultaneous and sequential subitizing are separate systems, and neither predicts math abilities. *Journal of Experimental Child Psychology, 178*, 86-103. https://doi.org/10.1016/j.jecp.2018.09.017

Ansari, D., Donlan, C., Thomas, M. S., Ewing, S. A., Peen, T., & Karmiloff-Smith, A. (2003). What makes counting count? Verbal and visuo-spatial contributions to typical and atypical number development. *Journal of Experimental Child Psychology, 85*(1), 50-62. https://doi.org/10.1016/S0022-0965(03)00026-2

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215-241. https://doi.org/10.1023/A:1024312321077

Auld, E., & Morris, P. (2019). The OECD and IELS: Redefining early childhood education for the 21st century. *Policy Futures in Education, 17*(1), 11-26. https://doi.org/10.1177/1478210318823949

Aulet, L. S., & Lourenco, S. F. (2018). The Developing Mental Number Line: Does Its Directionality Relate to 5- to 7-Year-Old Children's Mathematical Abilities? *Frontiers in psychology, 9*, 1142-1142. https://doi.org/10.3389/fpsyg.2018.01142

Aunio, P., & Niemivirta, M. (2010). Predicting children's mathematical performance in grade one by early numeracy. *Learning and Individual Differences, 20*(5), 427-435. https://doi.org/10.1016/j.lindif.2010.06.003

Baccaglini-Frank, A., & Maracci, M. (2015). Multi-touch technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education, 1*(1), 7-27. https://doi.org/10.1007/s40751-015-0002-4

Bahnmueller, J., Dresler, T., Ehlis, A.-C., Cress, U., & Nuerk, H.-C. (2014). NIRS in motion—unraveling the neurocognitive underpinnings of embodied numerical cognition. *Frontiers in Psychology, 5*(743). https://doi.org/10.3389/fpsyg.2014.00743

Bailey, D., Siegler, R., & Geary, D. (2014). Early predictors of middle school fraction knowledge. *Developmental science, 17*(5), 775-785. https://doi.org/10.1111/desc.12155

Bakar, K. A. (2017). Young Children's Representations of Addition in Problem Solving. *Creative Education, 8*(14), 2232. https://doi.org/10.4236/ce.2017.814153

Banegas, D. L., & Villacañas de Castro, L. S. (2015). A look at ethical issues in action research in education. *Argentinian Journal of Applied Linguistics, 2015, vol. 3, num. 1, p. 58-67*.

Barab, S., & Squire, K. (2004). Design-based research: Putting a stake in the ground. *The journal of the learning sciences, 13*(1), 1-14. https://doi.org/10.1207/s15327809jls1301_1

Barad, K. (2003). Posthumanist performativity: Toward an understanding of how matter comes to matter. *Signs: Journal of women in culture and society, 28*(3), 801-831. https://doi.org/10.1086/345321

Baroody, A. J. (1987). The development of counting strategies for single-digit addition. *Journal for Research in Mathematics Education*, 141-157. https://doi.org/10.2307/749248

Baroody, A. J. (2016). Curricular approaches to connecting subtraction to addition and fostering fluency with basic differences in grade 1. *PNA. Revista de Investigación en Didáctica de la Matemática, 10*(3), 161-190. https://doi.org/10.30827/pna.v10i3.6087

Barsalou, L. W. (1999). Perceptions of perceptual symbols. *Behavioral and brain sciences, 22*(04), 637-660. https://doi.org/10.1017/S0140525X99532147

Barsalou, L. W. (2003). Situated simulation in the human conceptual system. *Language and Cognitive Processes, 18*(5-6), 513-562. https://doi.org/10.1080/01690960344000026

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617-645. https://doi.org/10.1146/annurev.psych.59.103006.093639

Barsalou, L. W., Niedenthal, P. M., Barbey, A. K., & Ruppert, J. A. (2003). Social embodiment. *Psychology of Learning and Motivation, 43*, 43-92. https://doi.org/10.1016/S0079-7421(03)01011-9

Barsalou, L. W., Simmons, K. W., Barbey, A. K., & Wilson, C. D. (2003). Grounding conceptual knowledge in modality-specific systems. *Trends in Cognitive Sciences, 7*(2), 84. https://doi.org/10.1016/S1364-6613(02)00029-3

Barsalou, L. W., & Wiemer-Hastings, K. (2005). Situating abstract concepts. *Grounding cognition: The role of perception and action in memory, language, and thought*, 129-163. https://doi.org/10.1017/CBO9780511499968.007

Baxter, P., & Jack, S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The qualitative report, 13*(4), 544-559.

Beck, M. M., Lind, R. R., Geertsen, S. S., Ritz, C., Lundbye-Jensen, J., & Wienecke, J. (2016). Motor-Enriched Learning Activities Can Improve Mathematical Performance in Preadolescent Children. *Frontiers in Human Neuroscience, 10*(645). https://doi.org/10.3389/fnhum.2016.00645

Beilock, S. L., & Goldin-Meadow, S. (2010). Gesture Changes Thought by Grounding It in Action. *Psychological Science, 21*(11), 1605-1610. https://doi.org/10.1177/0956797610385353

Bermejo, V. (1996). Cardinality development and counting. *Developmental Psychology, 32*(2), 263. https://doi.org/10.1037/0012-1649.32.2.263

Bermejo, V., Morales, S., & Garcia deOsuna, J. (2004). Supporting children's development of cardinality understanding. *Learning and Instruction, 14*(4), 381-398. https://doi.org/10.1016/j.learninstruc.2004.06.010

Berteletti, I., & Booth, J. R. (2016). Finger representation and finger-based strategies in the acquisition of number meaning and arithmetic. In D. B. Berch, D. C. Geary, & K. M. Koepke (Eds.), *Development of mathematical cognition* (pp. 109-139). Academic Press. https://doi.org/10.1016/B978-0-12-801871-2.00005-8

Bjorklund, D. F., & Rosenblum, K. E. (2001). Children's use of multiple and variable addition strategies in a game context. *Developmental Science, 4*(2), 184-194. https://doi.org/10.1111/1467-7687.00164

Björklund, C. (2021). Children's awareness of numbers' part-whole relations when bridging through 10. Proceedings of MADIF 12. Sustainable mathematics education in a digitalized world, Växjö.

Björklund, C., Ekdahl, A.-L., & Runesson Kempe, U. (2021). Implementing a structural approach in preschool number activities. Principles of an intervention program reflected in learning. *Mathematical Thinking and Learning, 23*(1), 72-94. https://doi.org/10.1080/10986065.2020.1756027

Björklund, C., Kullberg, A., & Kempe, U. R. (2018). Structuring versus counting: critical ways of using fingers in subtraction. *Zdm*, 1-12. https://doi.org/10.1007/s11858-018-0962-0

Björklund, C., Marton, F., & Kullberg, A. (2021). What is to be learnt? Critical aspects of elementary arithmetic skills. *Educational Studies in Mathematics, 107*(2), 261-284. https://doi.org/10.1007/s10649-021-10045-0

Björklund, C., van den Heuvel-Panhuizen, M., & Kullberg, A. (2020). Research on early childhood mathematics teaching and learning. *Zdm, 52*(4), 607-619. https://doi.org/10.1007/s11858-020-01177-3

Bjørnebye, M. (2019). Pre-schoolers' ability to synchronise multiple representations of numerosity in embodiment of a counting-on-strategy. Eleventh Congress of the European Society for Research in Mathematics Education (CERME11),

Bjørnebye, M., Sigurjonsson, T., & Solbakken, T. (2017). Perception, cognition and measurement of verbal and non-verbal aspects of the cardinal concept in bodily-spatial interaction. Tenth Congress of the European Society for Research in Mathematics Education (CERME10), Dublin, Ireland.

Bobis, J. (2008). Early spatial thinking and the development of number sense. *Australian Primary Mathematics Classroom, 13*(3), 4.

Borghi, A. M., & Pecher, D. (2011). Introduction to the special topic embodied and grounded cognition. *Frontiers in Psychology, 2*. https://doi.org/10.3389/fpsyg.2011.00187

Braund, M. J. (2008). The structures of perception: an ecological perspective. *Kritike, 2*(1), 123-144. https://doi.org/10.3860/krit.v2i1.574

Broaders, S. C., Cook, S. W., Mitchell, Z., & Goldin-Meadow, S. (2007). Making children gesture brings out implicit knowledge and leads to learning. *Journal of Experimental Psychology: General, 136*(4), 539. https://doi.org/10.1037/0096-3445.136.4.539

Brown, A. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The journal of the learning sciences, 2*(2), 141-178. https://doi.org/10.1207/s15327809jls0202_2

Butterworth, B. (1999). *The mathematical brain*. Macmillan.

Butterworth, B. (2018). Low numeracy: From brain to education. In M. G. B. Bussi & H.-L. Sun (Eds.), *Building the Foundation: Whole Numbers in the Primary Grades. The 23rd ICMI study.* (pp. 477-488). Springer, Cham. https://doi.org/10.1007/978-3-319-63555-2_20

Butterworth, B., Varma, S., & Laurillard, D. (2011). Dyscalculia: from brain to education. *Science, 332*(6033), 1049-1053. https://doi.org/10.1126/science.1201536

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology, 105*(2), 380. https://doi.org/doi.org/10.1037/a0031084

Carey, S. (2004). Bootstrapping & the origin of concepts. *Daedalus, 133*(1), 59-68. https://doi.org/10.1162/001152604772746701

Caviola, S., Carey, E., Mammarella, I. C., & Szucs, D. (2017). Stress, Time Pressure, Strategy Selection and Math Anxiety in Mathematics: A Review of the Literature. *Frontiers in Psychology, 8*(1488). https://doi.org/10.3389/fpsyg.2017.01488

Chan, W. W. L., Au, T. K., & Tang, J. (2014). Strategic counting: A novel assessment of place-value understanding. *Learning and Instruction, 29*, 78-94. https://doi.org/10.1016/j.learninstruc.2013.09.001

Chan, W. W. L., & Wong, T. T.-Y. (2016). The underlying number–space mapping among kindergarteners and its relation with early numerical abilities. *Journal of Experimental Child Psychology, 148*, 35-50. https://doi.org/10.1016/j.jecp.2016.03.010

Chenail, R. J. (2012). Conducting Qualitative Data Analysis: Managing Dynamic Tensions within. *Qualitative Report, 17*, 4. https://doi.org/10.46743/2160-3715/2012.2724

Cheng, Y.-L., & Mix, K. S. (2014). Spatial Training Improves Children's Mathematics Ability. *Journal of Cognition and Development, 15*(1), 2-11. https://doi.org/10.1080/15248372.2012.725186

Cheng, Z.-J. (2012). Teaching young children decomposition strategies to solve addition problems: An experimental study. *The Journal of Mathematical Behavior, 31*(1), 29-47. https://doi.org/10.1016/j.jmathb.2011.09.002

Churchland, P. S. (1989). *Neurophilosophy: Toward a unified science of the mind-brain*. MIT press. https://doi.org/10.7551/mitpress/4952.001.0001

Cipora, K., Patro, K., & Nuerk, H.-C. (2018). Situated influences on spatial-numerical associations. In T. L. Hubbard (Ed.), *Spatial biases in perception and cognition*. Cambridge University Press. https://doi.org/10.1017/9781316651247

Clark, A. (1998). *Being there: Putting brain, body, and world together again*. MIT press. https://doi.org/10.7551/mitpress/1552.001.0001

Clarke, B., Cheeseman, J., & Clarke, D. (2006). The mathematical knowledge and understanding young children bring to school. *Mathematics Education Research Journal, 18*(1), 78-102. https://doi.org/10.1007/BF03217430

Clausen-May, T. (2005). *Teaching maths to pupils with different learning styles*. SAGE Publications Ltd. https://doi.org/10.4135/9781446212011

Clements, D. H., & Sarama, J. (2009). Learning trajectories in early mathematics–sequences of acquisition and teaching. *Encyclopedia of language and Literacy Development, 7*, 1-6.

Clements, D. H., & Sarama, J. (2011). Early childhood mathematics intervention. *Science, 333*(6045), 968-970. https://doi.org/10.1126/science.1204537

Clements, D. H., & Sarama, J. (2013). Rethinking early mathematics: What is research-based curriculum for young children? In L. English & J. Mulligan (Eds.), *Reconceptualizing Early Mathematics Learning. Advances in Mathematics Education* (pp. 121-147). Springer, Dordrecht. https://doi.org/10.1007/978-94-007-6440-8_7

Cohen, L., Manion, L., Morrison, K., & Bell, R. C. (2011). *Research methods in education*. Routledge.

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *The Journal of the learning sciences, 13*(1), 15-42. https://doi.org/10.1207/s15327809jls1301_2

Cook, S. W. (2011). Abstract thinking in space and time: Using gesture to learn math. *Cognitie, Creier, Comportament/Cognition, Brain, Behavior, 15*(4), 553-570.

Cook, S. W., Mitchell, Z., & Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition, 106*(2), 1047-1058. https://doi.org/10.1016/j.cognition.2007.04.010

Cook, S. W., Yip, T. K., & Goldin-Meadow, S. (2012). Gestures, but not meaningless movements, lighten working memory load when explaining math. *Language and Cognitive Processes, 27*(4), 594-610. https://doi.org/10.1080/01690965.2011.567074

Cooper , J. R. G. (1984). Early number development: Discovering number space with addition and subtraction. In C. Sophian (Ed.), *Origins of cognitive skills: The eighteenth annual Carnegie symposium on cognition* (pp. 157-192). Hillsdale, NJ: Erlbaum.

Cornu, V., Schiltz, C., Martin, R., & Hornung, C. (2018). Visuo-spatial abilities are key for young children's verbal number skills. *Journal of Experimental Child Psychology, 166*, 604-620. https://doi.org/10.1016/j.jecp.2017.09.006

Crollen, V., & Noël, M.-P. (2015). The role of fingers in the development of counting and arithmetic skills. *Acta Psychologica, 156*, 37-44. https://doi.org/10.1016/j.actpsy.2015.01.007

Dackermann, T., Fischer, U., Huber, S., Nuerk, H.-C., & Moeller, K. (2016). Training the equidistant principle of number line spacing. *Cognitive Processing, 17*(3), 243-258. https://doi.org/10.1007/s10339-016-0763-8

Dackermann, T., Fischer, U., Nuerk, H.-C., Cress, U., & Moeller, K. (2017). Applying embodied cognition: from useful interventions and their theoretical underpinnings to practical applications. *Zdm, 49*(4), 545-557. https://doi.org/10.1007/s11858-017-0850-z

De Smedt, B., Verschaffel, L., & Ghesquière, P. (2009). The predictive value of numerical magnitude comparison for individual differences in mathematics achievement. *Journal of Experimental Child Psychology, 103*(4), 469-479. https://doi.org/10.1016/j.jecp.2009.01.010

Dehaene, S. (2011). *The number sense: How the mind creates mathematics*. Oxford University Press.

Dehaene, S., Bossini, S., & Giraux, P. (1993). The mental representation of parity and number magnitude. *Journal of Experimental Psychology: General, 122*(3), 371.

Denzin, N. K., & Lincoln, Y. S. (2008). *Strategies of qualitative inquiry* (3 ed., Vol. 2). Sage.

Desoete, A., & Grégoire, J. (2006). Numerical competence in young children and in children with mathematics learning disabilities. *Learning and Individual Differences, 16*(4), 351-367. https://doi.org/10.1016/j.lindif.2006.12.006

DeSutter, D., & Stieff, M. (2017). Teaching students to think spatially through embodied actions: Design principles for learning environments in science, technology, engineering, and mathematics. *Cognitive research: principles and implications, 2*(1), 22. https://doi.org/10.1186/s41235-016-0039-y

DiSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *The journal of the learning sciences, 13*(1), 77-103. https://doi.org/10.1207/s15327809jls1301_4

Domahs, F., Moeller, K., Huber, S., Willmes, K., & Nuerk, H.-C. (2010). Embodied numerosity: implicit hand-based representations influence symbolic number processing across cultures. *Cognition, 116*(2), 251-266. https://doi.org/10.1016/j.cognition.2010.05.007

Donald, M. (2005). Imitation and Mimesis. In S. Hurley & N. Chater (Eds.), *Perspectives on Imitation: From neuroscience to social science* (Vol. 2, pp. 283-300). https://doi.org/10.7551/mitpress/5331.003.0017

Donnelly, J. E., Hillman, C. H., Castelli, D., Etnier, J. L., Lee, S., Tomporowski, P., Lambourne, K., & Szabo-Reed, A. N. (2016). Physical activity, fitness, cognitive function, and academic achievement in children: a systematic review. *Medicine and science in sports and exercise, 48*(6), 1197. https://doi.org/10.1249/MSS.0000000000000901

Dooley, L. M. (2002). Case study research and theory building. *Advances in developing human resources, 4*(3), 335-354. https://doi.org/10.1177/1523422302043007

Dowker, A. (2005). *Individual differences in arithmetic: Implications for psychology, neuroscience and education*. Psychology Press. https://doi.org/10.4324/9780203324899

Driscoll, D. L., Appiah-Yeboah, A., Salib, P., & Rupert, D. J. (2007). Merging qualitative and quantitative data in mixed methods research: How to and why not. *Ecological and Environmental Anthropology, 3(1)*, 19 - 28.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., Pagani, L. S., Feinstein, L., Engel, M., Brooks-Gunn, J., Sexton, H., Duckworth, K., Japel, C., & American Psychological Association, W. D. C. (2007). School Readiness and Later Achievement. *Developmental Psychology, 43*(6), 1428-1446. https://doi.org/10.1037/0012-1649.43.6.1428

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1), 103-131. https://doi.org/10.1007/s10649-006-0400-z

Eisenhardt, K. M., & Graebner, M. E. (2007). Theory building from cases: Opportunities and challenges. *Academy of management journal, 50*(1), 25-32.

Eisenhart, M. (2009). Generalization from qualitative inquiry. In K. Ercikan & W.-M. Roth (Eds.), *Generalizing from educational research* (pp. 51-66). Routledge.

Ekdahl, A.-L., Venkat, H., & Runesson, U. (2016). Coding teaching for simultaneity and connections. *Educational Studies in Mathematics, 93*(3), 293-313. https://doi.org/10.1007/s10649-016-9700-0

Ellemor-Collins, D., & Wright, R. (2009). Structuring numbers 1 to 20: Developing facile addition and subtraction. *Mathematics Education Research Journal, 21*(2), 50-75. https://doi.org/10.1007/BF03217545

Ellemor-Collins, D., & Wright, R. J. (2007). Assessing pupil knowledge of the sequential structure of number. *Educational and Child Psychology, 24*(2), 54-63.

Ellingson, L. L. (2011). Analysis and representation across the continuum. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (4 ed., pp. 595-610). Sage.

Erickson, K. I., Hillman, C. H., & Kramer, A. F. (2015). Physical activity, brain, and cognition. *Current opinion in behavioral sciences, 4*, 27-32. https://doi.org/10.1016/j.cobeha.2015.01.005

Feigenson, L., Dehaene, S., & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences, 8*(7), 307-314. https://doi.org/10.1016/j.tics.2004.05.002

Firestone, W. A. (1993). Alternative arguments for generalizing from data as applied to qualitative research. *Educational Researcher, 22*(4), 16-23. https://doi.org/10.3102/0013189X022004016

Fischer, B., Gebhardt, C., & Hartnegg, K. (2008). Subitizing and Visual Counting in Children with Problems in Acquiring Basic Arithmetic Skills. *Optometry & Vision Development, 39*(1).

Fischer, M. H., & Brugger, P. (2011). When digits help digits: spatial–numerical associations point to finger counting as prime example of embodied cognition. *Frontiers in psychology, 2*, 260. https://doi.org/10.3389/fpsyg.2011.00260

Fischer, U., Moeller, K., Bientzle, M., Cress, U., & Nuerk, H.-C. (2011). Sensori-motor spatial training of number magnitude representation. *Psychonomic Bulletin & Review, 18*(1), 177-183. https://doi.org/10.3758/s13423-010-0031-3

Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry, 12*(2), 219-245. https://doi.org/10.1177/1077800405284363

Foglia, L., & Wilson, R. A. (2013). Embodied Cognition. *Wiley Interdisciplinary Reviews: Cognitive Science, 4*. https://doi.org/10.1002/wcs.1226

Fosnot, C. T., & Dolk, M. (2001). *Young Mathematicians at Work: Constructing Number Sense, Addition, and Subtraction*. Heinemann

Freudenthal, H. (1986). *Didactical phenomenology of mathematical structures* (Vol. 1). Springer Science & Business Media.

Fuchs, L. S., Fuchs, D., Compton, D. L., Bryant, J. D., Hamlett, C. L., & Seethaler, P. M. (2007). Mathematics screening and progress monitoring at first grade: Implications for responsiveness to intervention. *Exceptional Children, 73*(3), 311-330. https://doi.org/10.1177/001440290707300303

Fugate, J. M. B., Macrine, S. L., & Cipriano, C. (2018). The role of embodied cognition for transforming learning. *International Journal of School & Educational Psychology*, 1-15. https://doi.org/10.1080/21683603.2018.1443856

Fuson, K. C. (1988). *Children's counting and concepts of number*. Springer-Verlag.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). Macmillan.

Fuster, J. M. (2009). Cortex and memory: emergence of a new paradigm. *Journal of cognitive neuroscience, 21*(11), 2047-2072. https://doi.org/10.1162/jocn.2009.21280

Gallagher, S., & Lindgren, R. (2015). Enactive metaphors: Learning through full-body engagement. *Educational Psychology Review, 27*(3), 391-404. https://doi.org/10.1007/s10648-015-9327-1

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive neuropsychology, 22*(3-4), 455-479. https://doi.org/10.1080/02643290442000310

Gallese, V., & Sinigaglia, C. (2011). What is so special about embodied simulation? *Trends in Cognitive Sciences, 15*(11), 512-519. https://doi.org/10.1016/j.tics.2011.09.003

Gallistel, C. R. (2011). Mental magnitudes. In S. Dehaene & E. Brannon (Eds.), *Space, time and number in the brain: Searching for the foundations of mathematical thought* (pp. 3-12). Elsevier. https://doi.org/10.1016/B978-0-12-385948-8.00001-3

Garvis, S., Harju-Luukkainen, H., & Yngvesson, T. (2019). Towards a test-driven early childhood education: Alternative practices to testing children. In *Compassion and empathy in educational contexts* (pp. 61-77). Springer. https://doi.org/https://doi.org/10.1007/978-3-030-18925-9_4

Geary, D. C., Bailey, D. H., Littlefield, A., Wood, P., Hoard, M. K., & Nugent, L. (2009). First-grade predictors of mathematical learning disability: A latent class trajectory analysis. *Cognitive Development, 24*(4), 411-429. https://doi.org/10.1016/j.cogdev.2009.10.001

Geary, D. C., Hoard, M. K., Byrd-Craven, J., & DeSoto, M. C. (2004). Strategy choices in simple and complex addition: Contributions of working memory and counting knowledge for children with mathematical disability. *Journal of Experimental Child Psychology, 88*(2), 121-151. https://doi.org/10.1016/j.jecp.2004.03.002

Gebuis, T., & Reynvoet, B. (2012). The role of visual information in numerosity estimation. *PLoS One, 7*(5), e37426. https://doi.org/10.1371/journal.pone.0037426

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Harvard University Press.

Gerring, J. (2007). Is there a (viable) crucial-case method? *Comparative Political Studies, 40*(3), 231-253. https://doi.org/10.1177/0010414006290784

Gersten, R., Jordan, N. C., & Flojo, J. R. (2005). Early identification and interventions for students with mathematics difficulties. *Journal of learning disabilities, 38*(4), 293-304. https://doi.org/10.1177/00222194050380040301

Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting, and knowing: toward an ecological psychology* (pp. 67-82). Lawrence Erlbaum.

Gibson, J. J. (1979). *The ecological approach to visual perception*. MA: Houghton Mifflin.

Giddens, A. (1979). *Central problems in social theory: Action, structure, and contradiction in social analysis* (Vol. 241). Univ of California Press. https://doi.org/10.1007/978-1-349-16161-4

Gilmore, C. K., & Bryant, P. (2006). Individual differences in children's understanding of inversion and arithmetical skill. *British Journal of Educational Psychology, 76*(2), 309-331. https://doi.org/10.1348/000709905X39125

Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. A. (2009). Gesturing Gives Children New Ideas About Math. *Psychological Science (Wiley-Blackwell), 20*(3), 267-272. https://doi.org/10.1111/j.1467-9280.2009.02297.x

Goldin-Meadow, S., Nusbaum, H., Kelly, S. D., & Wagner, S. (2001). Explaining math: Gesturing lightens the load. *Psychological science, 12*(6), 516-522. https://doi.org/10.1111/1467-9280.00395

Goodwin, C. (2000). Action and embodiment within situated human interaction. *Journal of Pragmatics, 32*(10), 1489-1522. https://doi.org/10.1016/S0378-2166(99)00096-X

Gordon, R., Chernyak, N., & Cordes, S. (2019). Get to the point: Preschoolers' spontaneous gesture use during a cardinality task. *Cognitive Development, 52*, 100818. https://doi.org/https://doi.org/10.1016/j.cogdev.2019.100818

Gray, E., Pitta, D., & Tall, D. (2000). Objects, actions, and images: A perspective on early number development. *The Journal of Mathematical Behavior, 18*(4), 401-413. https://doi.org/10.1016/S0732-3123(00)00025-0

Groen, G., & Parkman, J. M. (1972). A chronometric analysis of simple addition. *Psychological review, 79*(4), 329. https://doi.org/10.1037/h0032950

Guba, E. G. (1981). Criteria for assessing the trustworthiness of naturalistic inquiries. *Ectj, 29*(2), 75. https://doi.org/10.1007/BF02766777

Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2012). The relation between spatial skill and early number knowledge: The role of the linear number line. *Developmental Psychology, 48*(5), 1229-1241. https://doi.org/10.1037/a0027433

Gunderson, E. A., Spaepen, E., Gibson, D., Goldin-Meadow, S., & Levine, S. C. (2015). Gesture as a window onto children's number knowledge. *Cognition, 144*, 14-28. https://doi.org/10.1016/j.cognition.2015.07.008

Göbel, S. M., Shaki, S., & Fischer, M. H. (2011). The cultural number line: a review of cultural and linguistic influences on the development of number processing. *Journal of Cross-Cultural Psychology, 42*(4), 543-565. https://doi.org/10.1177/0022022111406251

Hannula, M. M. (2005). *Spontaneous focusing on numerosity in the development of early mathematical skills* [Doctoral Thesis], Turun Yliopisto.

Hannula, M. M., Lehtinen, E., Mattinen, A., & Räsänen, P. (2005). *Spontaneous Focusing on Numerosity in the Development of Early Mathematical Skills*. University of Turku.

Hannula, M. M., Rasanen, P., & Lehtinen, E. (2007). Development of Counting Skills: Role of Spontaneous Focusing on Numerosity and Subitizing-Based Enumeration. *Mathematical Thinking & Learning: An International Journal, 9*(1), 51-57. https://doi.org/0.1080/10986060709336605

Harmon-Jones, E., Amodio, D. M., & Harmon-Jones, C. (2009). Action-based model of dissonance: A review, integration, and expansion of conceptions of cognitive conflict. *Advances in experimental social psychology, 41*, 119-166. https://doi.org/10.1016/S0065-2601(08)00403-6

Harnad, S. (1990). The symbol grounding problem. *Physica D: Nonlinear Phenomena, 42*(1-3), 335-346. https://doi.org/10.1016/0167-2789(90)90087-6

Hawes, Z., Moss, J., Caswell, B., Naqvi, S., & MacKinnon, S. (2017). Enhancing Children's Spatial and Numerical Skills through a Dynamic Spatial Approach to Early Geometry Instruction: Effects of a 32-Week Intervention. *Cognition and Instruction, 35*(3), 236-264. https://doi.org/10.1080/07370008.2017.1323902

Heath, C., Hindmarsh, J., & Luff, P. (2010). *Video in qualitative research*. Sage Publications. https://doi.org/10.4135/9781526435385

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual–spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684. https://doi.org/10.1037/0022-0663.91.4.684

Herawaty, D., & Widada, W. (2017). The influence of contextual learning models and the cognitive conflict to understand mathematical concepts and problems solving abilities. 1st Annual International Conference on Mathematics, Science, and Education (ICoMSE 2017),

Heron, J., & Reason, P. (2006). The practice of co-operative inquiry: Research 'with'rather than 'on'people. *Handbook of action research*, 144-154.

Hostetter, A. B., & Alibali, M. W. (2019). Gesture as simulated action: Revisiting the framework. *Psychonomic Bulletin & Review, 26*(3), 721-752. https://doi.org/10.3758/s13423-018-1548-0

Houssart, J. (2013). *Low attainers in primary mathematics: The whisperers and the maths fairy*. Routledge. https://doi.org/10.4324/9780203464151

Hunting, R. P. (2003). Part-whole number knowledge in preschool children. *The Journal of Mathematical Behavior, 22*(3), 217-235. https://doi.org/10.1016/S0732-3123(03)00021-X

Hurley, S. L., & Chater, N. (2005). *Perspectives on Imitation: Imitation, human development, and culture* (Vol. 2). MIT Press.

Husserl, E. (1970). *The crisis of European sciences and transcendental phenomenology: An introduction to phenomenological philosophy*. Northwestern University Press. https://doi.org/10.1017/CBO9781139025935.001

Hutchins, E. (2006). The distributed cognition perspective on human interaction. *Roots of human sociality: Culture, cognition and interaction, 1*, 375.

Hyde, D. C., & Spelke, E. S. (2011). Neural signatures of number processing in human infants: evidence for two core systems underlying numerical cognition. *Developmental Science, 14*(2), 360-371. https://doi.org/10.1111/j.1467-7687.2010.00987.x

Ionescu, T., & Vasc, D. (2014). Embodied cognition: challenges for psychology and education. *Procedia-Social and Behavioral Sciences, 128*, 275-280. https://doi.org/10.1016/j.sbspro.2014.03.156

Jacobs, J. K., Kawanaka, T., & Stigler, J. W. (1999). Integrating qualitative and quantitative approaches to the analysis of video data on classroom teaching. *International Journal of Educational Research, 31*(8), 717-724. https://doi.org/10.1016/S0883-0355(99)00036-1

Jansen, B. R., Hofman, A. D., Straatemeier, M., Bers, B. M., Raijmakers, M. E., & Maas, H. L. (2014). The role of pattern recognition in children's exact enumeration of small numbers. *British Journal of Developmental Psychology, 32*(2), 178-194. https://doi.org/10.1111/bjdp.12032

Johnson-Glenberg, M. C. (2018). Immersive VR and education: Embodied design principles that include gesture and hand controls. *Frontiers in Robotics and AI, 5*, 81. https://doi.org/10.3389/frobt.2018.00081

Johnson-Glenberg, M. C., Birchfield, D. A., Tolentino, L., & Koziupa, T. (2014). Collaborative embodied learning in mixed reality motion-capture environments: Two science studies. *Journal of Educational Psychology, 106*(1), 86. https://doi.org/10.1037/a0034008

Jordan, K. E., & Brannon, E. M. (2006). The multisensory representation of number in infancy. *Proceedings of the National Academy of Sciences of the United States of America, 103*(9), 3486-3489. https://doi.org/10.1073/pnas.0508107103

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2008). Development of number combination skill in the early school years: When do fingers help? *Developmental Science, 11*(5), 662-668. https://doi.org/10.1111/j.1467-7687.2008.00715.x

Jung, M. (2011). Number relationships in preschool. *Teaching Children Mathematics, 17*(9), 550-557. https://doi.org/10.5951/teacchilmath.17.9.0550

Jung, M., Hartman, P., Smith, T., & Wallace, S. (2013). The effectiveness of teaching number relationships in preschool. *Online Submission, 6*(1), 165-178.

Kant, I., & Guyer, P. (1998). *Critique of pure reason*. Cambridge University Press. https://doi.org/10.1017/CBO9780511804649

Katzin, N., Cohen, Z. Z., & Henik, A. (2019). If it looks, sounds, or feels like subitizing, is it subitizing? A modulated definition of subitizing. *Psychonomic Bulletin & Review*. https://doi.org/10.3758/s13423-018-1556-0

Kaufman, E. L., Lord, M. W., Reese, T. W., & Volkmann, J. (1949). The discrimination of visual number. *The American journal of psychology, 62*(4), 498-525. https://doi.org/10.2307/1418556

Kennedy, M. M. (1979). Generalizing from single case studies. *Evaluation quarterly, 3*(4), 661-678. https://doi.org/10.1177/0193841X7900300409

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist, 41*(2), 75-86. https://doi.org/10.1207/s15326985ep4102_1

Klein, A., Starkey, P., Clements, D., Sarama, J., & Iyer, R. (2008). Effects of a pre-kindergarten mathematics intervention: A randomized experiment. *Journal of Research on Educational Effectiveness, 1*(3), 155-178. https://doi.org/10.1080/19345740802114533

Knoblauch, H., Schnettler, B., & Raab, J. (2012). *Video Analysis : Methodology and Methods : Qualitative Audiovisual Data Analysis in Sociology* (3 ed.). Peter Lang AG. https://doi.org/10.3726/978-3-653-02667-2

Krause, F., Meyer, M., Bekkering, H., Hunnius, S., & Lindemann, O. (2019). Interaction between perceptual and motor magnitudes in early childhood. *Cognitive Development, 49*, 11-19. https://doi.org/10.1016/j.cogdev.2018.11.001

Kroesbergen, E., Van Luit, J., Van Lieshout, E., Van Loosbroek, E., & Van de Rijt, B. (2009). Individual differences in early numeracy: The role of executive functions and subitizing. *Journal of Psychoeducational Assessment, 27*(3), 226-236. https://doi.org/10.1177/0734282908330586

Kullberg, A., Björklund, C., Brkovic, I., & Kempe, U. R. (2020). Effects of learning addition and subtraction in preschool by making the first ten numbers and their relations visible with finger patterns. *Educational Studies in Mathematics, 103*(2), 157-172. https://doi.org/10.1007/s10649-019-09927-1

Kullberg, A., Runesson Kempe, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *Zdm, 49*(4), 559-569. https://doi.org/10.1007/s11858-017-0858-4

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. University of Chicago press. https://doi.org/10.7208/chicago/9780226470993.001.0001

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh: The embodied mind and its challenge to western thought*. Basic Books.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being* (Vol. 6). Basic Books.

Landerl, K., Bevan, A., & Butterworth, B. (2004). Developmental dyscalculia and basic numerical capacities: A study of 8–9-year-old students. *Cognition, 93*(2), 99-125. https://doi.org/10.1016/j.cognition.2003.11.004

Le Corre, M., & Carey, S. (2007). One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles. *Cognition, 105*(2), 395-438. https://doi.org/10.1016/j.cognition.2006.10.005

Lee, M. D., & Sarnecka, B. W. (2010). A model of knower-level behavior in number concept development. *Cognitive Science, 34*(1), 51-67. https://doi.org/10.1111/j.1551-6709.2009.01063.x

LeFevre, J.-A., Jimenez Lira, C., Sowinski, C., Cankaya, O., Kamawar, D., & Skwarchuk, S.-L. (2013). Charting the role of the number line in mathematical development. *Frontiers in Psychology, 4*(641). https://doi.org/10.3389/fpsyg.2013.00641

Levine, S. C., Jordan, N. C., & Huttenlocher, J. (1992). Development of calculation abilities in young children. *Journal of Experimental Child Psychology, 53*(1), 72-103. https://doi.org/10.1016/S0022-0965(05)80005-0

Levine, S. C., Suriyakham, L. W., Rowe, M. L., Huttenlocher, J., & Gunderson, E. A. (2010). What counts in the development of young children's number knowledge? *Developmental psychology, 46*(5), 1309. https://doi.org/10.1037/a0019671

Lewis Presser, A., Clements, M., Ginsburg, H., & Ertle, B. (2015). Big Math for Little Kids: The Effectiveness of a Preschool and Kindergarten Mathematics Curriculum. *Early Education and Development, 26*(3), 399-426. https://doi.org/10.1080/10409289.2015.994451

Li, X., & Baroody, A. J. (2014). Young children's spontaneous attention to exact quantity and verbal quantification skills. *European Journal of Developmental Psychology, 11*(5), 608-623. https://doi.org/10.1080/17405629.2014.896788

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry* (Vol. 75). Sage. https://doi.org/10.1002/9781405165518.wbeosn006

Link, T., Moeller, K., Huber, S., Fischer, U., & Nuerk, H.-C. (2013). Walk the number line – An embodied training of numerical concepts. *Trends in Neuroscience and Education, 2*(2), 74-84. https://doi.org/10.1016/j.tine.2013.06.005

Lira, C. J., Carver, M., Douglas, H., & LeFevre, J.-A. (2017). The integration of symbolic and non-symbolic representations of exact quantity in preschool children. *Cognition, 166*, 382-397. https://doi.org/10.1016/j.cognition.2017.05.033

Lüken, M. M. (2012). Young Children's Structure Sense. *Journal für Mathematik-Didaktik, 33*(2), 263-285. https://doi.org/10.1007/s13138-012-0036-8

Malinverni, L., Pares, N., Malinverni, L., & Pares, N. (2014). Learning of abstract concepts through full-body interaction: A systematic review. *Educational Technology & Society, 17*(4), 100-116.

Mansfield, A., Cotton, W. G., & Ginns, P. (2020). Design principles of youth development programs in outdoor environments: a scoping review. *Journal of Outdoor and Environmental Education*. https://doi.org/10.1007/s42322-020-00063-6

Marton, F. (2015). *Necessary conditions of learning*. Routledge. https://doi.org/10.4324/9781315816876

Maxwell, J. A. (2008). Designing a qualitative study. *The SAGE Handbook of Applied Social Research Methods, SAGE, Los Angeles*, 214-246. https://doi.org/10.4135/9781483348858.n7

McClelland, M. M., Acock, A. C., & Morrison, F. J. (2006). The impact of kindergarten learning-related skills on academic trajectories at the end of elementary school. *Early Childhood Research Quarterly, 21*(4), 471-490. https://doi.org/10.1016/j.ecresq.2006.09.003

McCluskey, C., Mulligan, J., & Van Bergen, P. (2018). Noticing Mathematical Pattern and Structure Embodied in Young Children's Play. In J. Hunter, L. Darragh, & P. Perger (Eds.), *Making waves, opening spaces: proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 535-542). MERGA.

McCrink, K., & Opfer, J. E. (2014). Development of spatial-numerical associations. *Current directions in psychological science, 23*(6), 439-445. https://doi.org/10.1177/0963721414549751

McGuire, P., Kinzie, M., & Berch, D. (2012). Developing Number Sense in Pre-K with Five-Frames. *Early Childhood Education Journal, 40*(4), 213-222. https://doi.org/10.1007/s10643-011-0479-4

Merleau-Ponty, M. (2005). *Phenomenology of perception*. Routledge. Retrieved from http://site.ebrary.com/id/5006132

Mintzberg, H. (2017). Developing theory about the development of theory. In S. W. Floyd & B. Wooldridge (Eds.), *Handbook of Middle Management Strategy Process Research* (pp. 177-196). Edward Elgar Publishing. https://doi.org/10.4337/9781783473250.00017

Mix, K. S., Sandhofer, C. M., Moore, J. A., & Russell, C. (2012). Acquisition of the cardinal word principle: The role of input. *Early Childhood Research Quarterly, 27*(2), 274-283. https://doi.org/10.1016/j.ecresq.2011.10.003

Mockler, N. (2014). When 'research ethics' become 'everyday ethics': The intersection of inquiry and practice in practitioner research. *Educational Action Research, 22*(2), 146-158. https://doi.org/10.1080/09650792.2013.856771

Moeller, K., Fischer, U., Link, T., Wasner, M., Huber, S., Cress, U., & Nuerk, H.-C. (2012). Learning and development of embodied numerosity. *Cognitive processing, 13*(1), 271-274. https://doi.org/10.1007/s10339-012-0457-9

Moeller, K., Martignon, L., Wessolowski, S., Engel, J., & Nuerk, H.-C. (2011). Effects of finger counting on numerical development – The opposing views of neurocognition and mathematics education. *Frontiers in Psychology, 2*(328). https://doi.org/10.3389/fpsyg.2011.00328

Moser, T., & Martinsen, M. T. (2010). The outdoor environment in Norwegian kindergartens as pedagogical space for toddlers' play, learning and development. *European Early Childhood Education Research Journal, 18*(4), 457-471. https://doi.org/10.1080/1350293X.2010.525931

Mulligan, J. (2010). *Reconceptualising early mathematics learning* Teaching Mathematics? Make it count. Retrieved from https://research.acer.edu.au/research_conference/RC2010/16august/13

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33-49. https://doi.org/10.1007/BF03217544

Mulligan, J., & Mitchelmore, M. (2013). Early awareness of mathematical pattern and structure. In L. D. English & J. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 29-45). Springer. https://doi.org/10.1007/978-94-007-6440-8_3

Mulligan, J., Mitchelmore, M., & Prescott, A. (2006). Integrating concepts and processes in early mathematics: The Australian Pattern and Structure Awareness Project (PASMAP). Conference of the International Group for the Psychology of Mathematics Education,

Mulligan, J., & Vergnaud, G. (2006). Research on children's early mathematical development: Towards integrated perspectives. In *Handbook of research on the psychology of mathematics education* (pp. 117-146). Brill Sense.

Nathan, M. J. (2008). An embodied cognition perspective on symbols, gesture, and grounding instruction. *Symbols and embodiment: Debates on meaning and cognition, 18*, 375-396. https://doi.org/10.1093/acprof:oso/9780199217274.003.0018

NESH (The National Committee for Research Ethics in the Social Sciences and the Humanities, L. a. T. (2016). *Forskningsetiske retningslinjer [Guidelines for research ethics]*. Oslo. Retrieved from https://www.forskningsetikk.no/

Neuman, D. (2013). Att ändra arbetssätt och kultur inom den inledande aritmetikundervisningen. *Nordic Studies in Mathematics Education, 18*(2), 3-46.

Newcombe, N. S., Levine, S. C., & Mix, K. S. (2015). Thinking about quantity: the intertwined development of spatial and numerical cognition. *WIREs Cognitive Science, 6*(6), 491-505. https://doi.org/https://doi.org/10.1002/wcs.1369

Newman, S. D. (2016). Does finger sense predict addition performance? *Cognitive processing, 17*(2), 139-146. https://doi.org/10.1007/s10339-016-0756-7

Nicoladis, E., Pika, S., & Marentette, P. (2010). Are number gestures easier than number words for preschoolers? *Cognitive Development, 25*(3), 247-261. https://doi.org/10.1016/j.cogdev.2010.04.001

Nieder, A., & Dehaene, S. (2009). Representation of number in the brain. *Annual review of neuroscience, 32*, 185-208. https://doi.org/10.1146/annurev.neuro.051508.135550

Noël, M.-P. (2005). Finger gnosia: a predictor of numerical abilities in children? *Child Neuropsychology, 11*(5), 413-430. https://doi.org/10.1080/09297040590951550

Norwegian Ministry of Education and Research. (2017). *Framework Plan for the Kindergartens - contents and tasks*. Oslo. Retrieved from https://www.udir.no/

Norwegian Ministry of Education and Research. (2020). *Curriculum for Mathematics Year 1–10 (MAT01-05)*. Oslo. Retrieved from https://www.udir.no/

Novack, M. A., Congdon, E. L., Hemani-Lopez, N., & Goldin-Meadow, S. (2014). From action to abstraction: using the hands to learn math. *Psychological science, 25*(4), 903-910. https://doi.org/10.1177/0956797613518351

Nunes, T., Bryant, P., Barros, R., & Sylva, K. (2012). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology, 82*(1), 136-156. https://doi.org/10.1111/j.2044-8279.2011.02033.x

Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. *British Journal of Developmental Psychology, 25*(1), 147-166. https://doi.org/10.1348/026151006X153127

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*(1-3), 45-65. https://doi.org/10.1023/A:1003759711966

Obersteiner, A., Reiss, K., & Ufer, S. (2013). How training on exact or approximate mental representations of number can enhance first-grade students' basic number processing and arithmetic skills. *Learning and Instruction, 23*, 125-135. https://doi.org/10.1016/j.learninstruc.2012.08.004

OECD. (2017). *Starting Strong V: Transitions from Early Childhood Education and Care to Primary Education* https://doi.org/10.1787/9789264276253-en

Onwuegbuzie, A. J., & Leech, N. L. (2007). Sampling designs in qualitative research: Making the sampling process more public. *Qualitative Report, 12*(2), 238-254.

Ostad, S. A. (1997). Developmental differences in addition strategies: A comparison of mathematically disabled and mathematically normal children. *British Journal of Educational Psychology, 67*(3), 345-357. https://doi.org/10.1111/j.2044-8279.1997.tb01249.x

Ostad, S. A. (1998). Developmental differences in solving simple arithmetic word problems and simple number-fact problems: A comparison of mathematically normal and mathematically disabled children. *Mathematical Cognition, 4*(1), 1-19. https://doi.org/10.1080/135467998387389

Ostad, S. A., & Sorensen, P. M. (2007). Private speech and strategy-use patterns. *Journal of Learning Disabilities, 40*(1), 2-14. https://doi.org/10.1177/00222194070400010101

Paliwal, V., & Baroody, A. J. (2018). How best to teach the cardinality principle? *Early Childhood Research Quarterly, 44*, 152-160. https://doi.org/10.1016/j.ecresq.2018.03.012

Pande, P. (2020). Learning and expertise with scientific external representations: an embodied and extended cognition model. *Phenomenology and the Cognitive Sciences*. https://doi.org/10.1007/s11097-020-09686-y

Parsons, S., & Bynner, J. (2005). *Does numeracy matter more?* National Research and Development Centre for adult literacy and numeracy. Institute of Education.

Patro, K., Nuerk, H.-C., Cress, U., & Haman, M. (2014). How number-space relationships are assessed before formal schooling: A taxonomy proposal. *Frontiers in Psychology, 5*(419). https://doi.org/10.3389/fpsyg.2014.00419

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3 ed.). Thousand Oaks. Sage Publications.

Penner-Wilger, M., Fast, L., LaFevre, J.-A., Smith-Chant, B. L., Skwarchuck, S.-L., Kamawar, D., & Bisanz, J. (2007). The foundations of numeracy: Subitizing, finger gnosia, and fine motor ability. Proceedings of the Annual Meeting of the Cognitive Science Society,

Pesce, C. (2012). Shifting the focus from quantitative to qualitative exercise characteristics in exercise and cognition research. *Journal of Sport and Exercise Psychology, 34*(6), 766-786. https://doi.org/10.1123/jsep.34.6.766

Pexman, P. M. (2019). The role of embodiment in conceptual development. *Language, Cognition and Neuroscience, 34*(10), 1274-1283. https://doi.org/10.1080/23273798.2017.1303522

Piaget, J. (2001). *The psychology of intelligence*. Routledge. https://doi.org/10.4324/9780203164730

Piaget, J., Gattegno, C., & Hodgson, F. M. (1952). *The child's conception of number*. Routledge & Paul.

Piazza, M. (2010). Neurocognitive start-up tools for symbolic number representations. *Trends in Cognitive Sciences, 14*(12), 542-551. https://doi.org/10.1016/B978-0-12-385948-8.00017-7

Pinhas, M., & Fischer, M. H. (2008). Mental movements without magnitude? A study of spatial biases in symbolic arithmetic. *Cognition, 109*(3), 408-415. https://doi.org/10.1016/j.cognition.2008.09.003

Pixner, S., Kraut, C., & Dresen, V. (2017). Early Predictors for Basic Numerical and Magnitude Competencies in Preschool Children - Are They the Same or Different regarding Specific Subgroups? *Psychology, Vol.08No.02*, 16, Article 73739. https://doi.org/10.4236/psych.2017.82016

Plomp, T. (2013). Educational Design Research: An Introduction. In T. Plomp & N. Nieveen (Eds.), *Educational Design Research. Part A: An introduction* (pp. 10-51). Springer Science & Business Media. Retrieved from http://international.slo.nl/publications/edr/

Pontecorvo, C., & Sterponi, L. (2002). Learning to argue and reason through discourse in educational settings. In G. Wells & G. Claxton (Eds.), *Learning for life in the 21st century: Sociocultural perspectives on the future of education* (pp. 127-140). Blackwell. https://doi.org/10.1002/9780470753545

Posid, T., & Cordes, S. (2019). The effect of multimodal information on children's numerical judgments. *Journal of Experimental Child Psychology, 182*, 166-186. https://doi.org/https://doi.org/10.1016/j.jecp.2019.01.003

Pramling, N., Wallerstedt, C., Lagerlöf, P., Björklund, C., Kultti, A., Palmér, H., Magnusson, M., Thulin, S., Jonsson, A., & Pramling Samuelsson, I. (2019). *Play-Responsive Teaching in Early Childhood Education*. Springer Nature. https://doi.org/10.1007/978-3-030-15958-0

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM, 47*(6), 877-891. https://doi.org/10.1007/s11858-015-0722-3

Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). Semiotics in Theory and Practice in Mathematics Education. In *Semiotics in Mathematics Education* (pp. 5-29). Springer. https://doi.org/10.1007/978-3-319-31370-2

Prusak, N., & Hershkowitz, R. (2019). Task designed for promoting the concept of area in a problem solving environment-the story of one activity. International Symposium Elementary Mathematics Teaching,

Purpura, D. J., Baroody, A. J., & Lonigan, C. J. (2013). The transition from informal to formal mathematical knowledge: Mediation by numeral knowledge. *Journal of Educational Psychology, 105*(2), 453. https://doi.org/10.1037/a0031753

Purpura, D. J., & Lonigan, C. J. (2015). Early numeracy assessment: The development of the preschool early numeracy scales. *Early education and development, 26*(2), 286-313. Retrieved from http://www.tandfonline.com/doi/pdf/10.1080/10409289.2015.991084?needAccess=true

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA, 4*(2). https://doi.org/10.30827/pna.v4i2.6169

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of research in mathematics education, 2*(1), 7-44. https://doi.org/10.4471/redimat.2013.19

Radford, L. (2015). Rhythm as an integral part of mathematical thinking. In M. Bockarova, M. Danesi, D. Martinovic, & R. Núñez (Eds.), *Mind in mathematics: Essays on mathematical cognition and mathematical method* (pp. 68-85). LINCOM GmbH.

Radford, L., Arzarello, F., Edwards, L., & Sabena, C. (2017). The multimodal material mind: Embodiment in mathematics education.

Ramani, G., & Siegler, R. (2011). Reducing the gap in numerical knowledge between low-and middle-income preschoolers. *Journal of Applied Developmental Psychology, 32*(3), 146-159. https://doi.org/10.1016/j.appdev.2011.02.005

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., De Smedt, B., & Verschaffel, L. (2016). Spontaneous focusing on numerosity: A review of recent research. *Mediterranean Journal for Research in Mathematics Education, 15*, 1-25.

Reeve, R., Reynolds, F., Humberstone, J., & Butterworth, B. (2012). Stability and change in markers of core numerical competencies. *Journal of Experimental Psychology: General, 141*(4), 649. https://doi.org/10.1037/a0027520

Reikerås, E. (2016). Central skills in toddlers' and pre-schoolers' mathematical development, observed in play and everyday activities. *Nordic Studies in Mathematics Education, 21(4)*, 57-77.

Reikerås, E. (2020). Relations between play skills and mathematical skills in toddlers. *Zdm, 52*(4), 703-716. https://doi.org/10.1007/s11858-020-01141-1

Reikerås, E., Loge, I. K., & Knivsberg, A.-M. (2012). The Mathematical Competencies of Toddlers Expressed in Their Play and Daily Life Activities in Norwegian Kindergartens. *International Journal of Early Childhood, 44*(1), 91-114. https://doi.org/10.1007/s13158-011-0050-x

Reikerås, E., Moser, T., & Tønnessen, F. E. (2017). Mathematical skills and motor life skills in toddlers: do differences in mathematical skills reflect differences in motor skills? *European Early Childhood Education Research Journal, 25*(1), 72-88. https://doi.org/10.1080/1350293X.2015.1062664

Reikerås, E., & Salomonsen, T. (2019). Weak mathematical skills at an early age: persistent or temporary? Children with weak mathematical skills and their development from toddlers to preschoolers. *Early Child Development and Care, 189*(4), 670-682. https://doi.org/10.1080/03004430.2017.1337753

Riggs, K. J., Ferrand, L., Lancelin, D., Fryziel, L., Dumur, G., & Simpson, A. (2006). Subitizing in tactile perception. *Psychological Science, 17*(4), 271-272. https://doi.org/10.1111/j.1467-9280.2006.01696.x

Risko, E. F., & Gilbert, S. J. (2016). Cognitive offloading. *Trends in Cognitive Sciences, 20*(9), 676-688. https://doi.org/10.1016/j.tics.2016.07.002

Rizzolatti, G., & Craighero, L. (2004). The mirror-neuron system. *Annu. Rev. Neurosci., 27*, 169-192. https://doi.org/10.1146/annurev.neuro.27.070203.144230

Robinson, J. E., & Norris, N. F. (2001). Generalisation: the linchpin of evidence-based practice? *Educational Action Research, 9*(2), 303-310. https://doi.org/10.1080/09650790100200155

Rodríguez, J., Martí, E., & Salsa, A. (2018). Symbolic representations and cardinal knowledge in 3- and 4-year-old children. *Cognitive Development, 48*, 235-243. https://doi.org/10.1016/j.cogdev.2018.09.004

Roth, W.-M. (2009). *Mathematical representation at the interface of body and culture*. Information Age Publishing.

Roth, W.-M. (2012). Tracking the origins of signs in mathematical activity: A material phenomenological approach. *Cognitive science and interdisciplinary approaches to mathematical cognition*, 182-215.

Rowlands, M. (2010). *The new science of the mind: From extended mind to embodied phenomenology*. Mit Press. https://doi.org/10.7551/mitpress/9780262014557.001.0001

Ruddin, L. P. (2006). You can generalize stupid! Social scientists, Bent Flyvbjerg, and case study methodology. *Qualitative Inquiry, 12*(4), 797-812. https://doi.org/10.1177/1077800406288622

Rugani, R., Betti, S., Ceccarini, F., & Sartori, L. (2017). Act on Numbers: Numerical Magnitude Influences Selection and Kinematics of Finger Movement. *Frontiers in Psychology, 8*(1481). https://doi.org/10.3389/fpsyg.2017.01481

Ruiter, M., Loyens, S., & Paas, F. (2015). Watch Your Step Children! Learning Two-Digit Numbers Through Mirror-Based Observation of Self-Initiated Body Movements. *Educational Psychology Review, 27*(3), 457-474. https://doi.org/10.1007/s10648-015-9324-4

Russell, B. (1917). *Mysticism and logic*. George Allen & Unwin Ltd. https://doi.org/10.5962/bhl.title.19230

Räsänen, P., Salminen, J., Wilson, A. J., Aunio, P., & Dehaene, S. (2009). Computer-assisted intervention for children with low numeracy skills. *Cognitive Development, 24*(4), 450-472. https://doi.org/10.1016/j.cogdev.2009.09.003

Salmah, U., & Putri, R. I. I. (2015). Ten-Structure as Strategy of Addition 1-20 by Involving Spatial Structuring Ability for First Grade Students. *International Education Studies, 8*(11), 16-25. https://doi.org/10.5539/ies.v8n11p16

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the learning sciences, 23*(1), 18-36. https://doi.org/10.1080/10508406.2013.778204

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge. https://doi.org/10.4324/9780203883785

Sarnecka, B. W., & Carey, S. (2006). The development of human conceptual representations. In M. Johnson & Y. Munakata (Eds.), *Processes of Change in Brain and Cognitive Development: Attention and Performance XX* (pp. 473-496). https://doi.org/10.1002/acp.1330

Sarnecka, B. W., & Wright, C. E. (2013). The idea of an exact number: Children's understanding of cardinality and equinumerosity. *Cognitive Science, 37*(8), 1493-1506. https://doi.org/10.1111/cogs.12043

Sasanguie, D., Van den Bussche, E., & Reynvoet, B. (2012). Predictors for Mathematics Achievement? Evidence from a Longitudinal Study. *Mind, Brain, and Education, 6*(3), 119-128. https://doi.org/10.1111/j.1751-228X.2012.01147.x

Schaeffer, B., Eggleston, V. H., & Scott, J. L. (1974). Number development in young children. *Cognitive Psychology, 6*(3), 357-379. https://doi.org/10.1016/0010-0285(74)90017-6

Schneider, M., Beeres, K., Coban, L., Merz, S., Susan Schmidt, S., Stricker, J., & De Smedt, B. (2017). Associations of non-symbolic and symbolic numerical magnitude processing with mathematical competence: A meta-analysis. *Developmental science, 20*(3), e12372. https://doi.org/10.1111/desc.12372

Schneider, M., Merz, S., Stricker, J., De Smedt, B., Torbeyns, J., Verschaffel, L., & Luwel, K. (2018). Associations of number line estimation with mathematical competence: A meta-analysis. *Child development, 89*(5), 1467-1484. https://doi.org/10.1111/cdev.13068

Schwarz, B., & Prusak, N. (2016). The importance of multi-modality in mathematical argumentation. *The psychology of argument*, 387-406.

Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. Basic books. https://doi.org/10.4324/9781315237473

Schöner, P., & Benz, C. (2017). "Two, three and two more equals seven"–Preschoolers' perception and use of structures in sets. Tenth Congress of the European Society for Research in Mathematics Education (CERME10), Dublin, Ireland.

Sedaghatjou, M., & Campbell, S. R. (2017). Exploring cardinality in the era of touchscreen-based technology. *International Journal of Mathematical Education in Science and Technology, 48*(8), 1225-1239. https://doi.org/10.1080/0020739X.2017.1327089

Segal, A. (2011). *Do gestural interfaces promote thinking? Embodied interaction: Congruent gestures and direct touch promote performance in math* [Doctoral Thesis, Columbia University].

Segal, A., Tversky, B., & Black, J. (2014). Conceptually congruent actions can promote thought. *Journal of Applied Research in Memory and Cognition, 3*(3), 124-130. https://doi.org/10.1016/j.jarmac.2014.06.004

Shaki, S., & Fischer, M. H. (2012). Multiple spatial mappings in numerical cognition. *Journal of Experimental Psychology: Human Perception and Performance, 38*(3), 804. https://doi.org/10.1037/a0027562

Shaki, S., & Fischer, M. H. (2014). Random walks on the mental number line. *Experimental brain research, 232*(1), 43-49. https://doi.org/10.1007/s00221-013-3718-7

Shapiro, L. (2011). *Embodied cognition*. Routledge Press.

Shapiro, L. (2014). *The Routledge handbook of embodied cognition*. Routledge. https://doi.org/10.4324/9781315775845

Shapiro, L., & Stolz, S. (2019). Embodied cognition and its significance for education. *Theory and Research in Education, 17*, 19-39. https://doi.org/10.1177/1477878518822149

Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology, 101*(1), 88. https://doi.org/10.1037/a0013156

Shoval, E. (2011). Using mindful movement in cooperative learning while learning about angles. *Instructional Science, 39*(4), 453-466. https://doi.org/10.1007/s11251-010-9137-2

Shrager, J., & Siegler, R. (1998). SCADS: A model of children's strategy choices and strategy discoveries. *Psychological science, 9*(5), 405-410. https://doi.org/10.1111/1467-9280.00076

Siegler, R. (1998). *Emerging minds: The process of change in children's thinking*. Oxford University Press. https://doi.org/10.1093/oso/9780195077872.001.0001

Siegler, R., & Ramani, G. (2009). Playing linear number board games—but not circular ones—improves low-income preschoolers' numerical understanding. *Journal of Educational Psychology, 101*(3), 545. https://doi.org/10.1037/a0014239

Siegler, R., & Shrager, J. (1984). Strategy choice in addition and subtraction: How do children know what to do? In C. Sophian (Ed.), *Origins of cognitive skills* (pp. 229-293). NJ: Erlbaum.

Silverman, D. (2010). *Qualitative research* (3 ed.). Sage.

Silverman, D. (2013). *Doing qualitative research* (4 ed.). SAGE publications.

Silverman, D. (2014). *Interpreting qualitative data* (5 ed.). Sage Publications.

Sjøberg, S. (2018). The power and paradoxes of PISA: Should Inquiry-Based Science Education be sacrificed to climb on the rankings? *Nordic Studies in Science Education, 14*(2), 186-202. https://doi.org/doi.org/10.5617/nordina.6185

Skoura, X., Vinter, A., & Papaxanthis, C. (2009). Mentally Simulated Motor Actions in Children. *Developmental Neuropsychology, 34*(3), 356-367. https://doi.org/10.1080/87565640902801874

Skulmowski, A., Pradel, S., Kühnert, T., Brunnett, G., & Rey, G. D. (2016). Embodied learning using a tangible user interface: The effects of haptic perception and selective pointing on a spatial learning task. *Computers & Education, 92-93*, 64-75. https://doi.org/https://doi.org/10.1016/j.compedu.2015.10.011

Skulmowski, A., & Rey, G. D. (2018). Embodied learning: introducing a taxonomy based on bodily engagement and task integration. *Cognitive Research: Principles and Implications, 3*. https://doi.org/10.1186/s41235-018-0092-9

Spelke, E. S., & Kinzler, K. D. (2007). Core knowledge. *Developmental science, 10*(1), 89-96. https://doi.org/10.1111/j.1467-7687.2007.00569.x

Spelke, E. S., & Lee, S. A. (2012). Core systems of geometry in animal minds. *Philosophical Transactions of the Royal Society B: Biological Sciences, 367*(1603), 2784-2793. https://doi.org/10.1098/rstb.2012.0210

Sriraman, B., & Wu, K. (2020). Embodied Cognition. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 266-268). Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_179

Starkey, P., & Cooper, R. G. (1995). The development of subitizing in young children. *British Journal of Developmental Psychology, 13*(4), 399-420. https://doi.org/10.1111/j.2044-835X.1995.tb00688.x

Starkey, P., Klein, A., & Wakeley, A. (2004). Enhancing young children's mathematical knowledge through a pre-kindergarten mathematics intervention. *Early Childhood Research Quarterly, 19*(1), 99-120. https://doi.org/10.1016/j.ecresq.2004.01.002

Stevens, R. (2012). The missing bodies of mathematical thinking and learning have been found. *Journal of the Learning Sciences, 21*(2), 337-346. https://doi.org/10.1080/10508406.2011.614326

Streeck, J., Goodwin, C., & LeBaron, C. (2011). *Embodied interaction: Language and body in the material world*. Cambridge University Press.

Strike, K. (2006). The ethics of educational research. *Handbook of complementary methods in education research*, 57-73.

Tate, T. L., Thompson, R. H., & McKerchar, P. M. (2005). Training Teachers in an Infant Classroom to Use Embedded Teaching Strategies. *Education and Treatment of Children, 28*(3), 206-221. Retrieved from http://www.jstor.org.ezproxy.inn.no/stable/42899845

Thevenot, C., Barrouillet, P., Castel, C., & Uittenhove, K. (2016). Ten-year-old children strategies in mental addition: A counting model account. *Cognition, 146*, 48-57. https://doi.org/10.1016/j.cognition.2015.09.003

Threlfall, J. (2009). Strategies and flexibility in mental calculation. *Zdm, 41*(5), 541-555. https://doi.org/10.1007/s11858-009-0195-3

Thurmond, V. A. (2001). The point of triangulation. *Journal of nursing scholarship, 33*(3), 253-258. https://doi.org/10.1111/j.1547-5069.2001.00253.x

Torbeyns, J., Gilmore, C., & Verschaffel, L. (2015). The Acquisition of Preschool Mathematical Abilities: Theoretical, Methodological and Educational Considerations. *Mathematical Thinking and Learning, 17*(2-3), 99-115. https://doi.org/10.1080/10986065.2015.1016810

Tran, C., Smith, B., & Buschkuehl, M. (2017). Support of mathematical thinking through embodied cognition: Nondigital and digital approaches. *Cognitive research: principles and implications, 2*(1), 16. https://doi.org/10.1186/s41235-017-0053-8

Tucker, S. I., & Johnson, T. N. (2018). Development of Embodied Subitizing During Interactions with a Multi-Touch Digital Game. In *EdMedia+ Innovate Learning* (pp. 1236-1241). Retrieved from http://www.learntechlib.org/p/184334/

Tudge, J., Li, L., & Stanley, T. K. (2008). The impact of method on assessing young children's everyday mathematical experiences. *Contemporary perspectives on mathematics in early childhood education*, 187-214.

Valkenborghs, S. R., Noetel, M., Hillman, C. H., Nilsson, M., Smith, J. J., Ortega, F. B., & Lubans, D. R. (2019). The impact of physical activity on brain structure and function in youth: a systematic review. *Pediatrics, 144*(4). https://doi.org/10.1542/peds.2018-4032

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (Eds.). (2006). *Educational design research*. Routledge. https://doi.org/10.4324/9780203088364.

Van den Heuvel-Panhuizen, M. (2008). *Children learn mathematics: A learning-teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school*. BRILL. https://doi.org/10.1163/9789087903954

van Nes, F., & van Eerde, D. (2010). Spatial structuring and the development of number sense: A case study of young children working with blocks. *The Journal of Mathematical Behavior, 29*(3), 145-159. https://doi.org/10.1016/j.jmathb.2010.08.001

van Oers, B. (2010). Emergent mathematical thinking in the context of play. *Educational Studies in Mathematics, 74*(1), 23-37. https://doi.org/10.1007/s10649-009-9225-x

Varzi, A. C. (2007). Spatial reasoning and ontology: Parts, wholes, and locations. In M. Aiello, I. Pratt-Hartmann, & J. V. Benthem (Eds.), *Handbook of spatial logics* (pp. 945-1038). Springer. https://doi.org/https://doi.org/10.1007/978-1-4020-5587-4_15

Vazou, S., Pesce, C., Lakes, K., & Smiley-Oyen, A. (2019). More than one road leads to Rome: a narrative review and meta-analysis of physical activity intervention effects on cognition in youth. *International Journal of Sport and Exercise Psychology, 17*(2), 153-178. https://doi.org/10.1080/1612197X.2016.1223423

Venkat, H., Askew, M., Watson, A., Mason, J., Venkatakrishnan, H., Askew, M., & Mason, J. (2019). Architecture of mathematical structure. *For the Learning of Mathematics, 39*(1), 13-17.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557 - 628). Information Age Publishing.

Wang, Z., & Shah, P. (2014). The effect of pressure on high- and low-working-memory students: An elaboration of the choking under pressure hypothesis. *British Journal of Educational Psychology, 84*(2), 226-238. https://doi.org/doi.org/10.1111/bjep.12027

Waxer, M., & Morton, J. B. (2012). Cognitive conflict and learning. In N. M. Seel (Ed.), *Encyclopedia of the Sciences of Learning* (pp. 585-587). Springer US. https://doi.org/10.1007/978-1-4419-1428-6_280

Wender, K. F., & Rothkegel, R. (2000). Subitizing and its subprocesses. *Psychological Research, 64*(2), 81-92. https://doi.org/10.1007/s004260000021

Wertsch, J. V., & Wertsch, J. V. (1993). *Voices of the mind: Sociocultural approach to mediated action*. Harvard University Press. Retrieved from https://epdf.pub/voices-of-the-mind-sociocultural-approach-to-mediated-action.html

Whitacre, I., Hohensee, C., & Nemirovksy, R. (2009). Expressiveness and mathematics learning. In W.-M. Roth (Ed.), *Mathematical representation at the interface of body and culture* (pp. 275-308). IAP.

Whyte, J. C., & Bull, R. (2008). Number games, magnitude representation, and basic number skills in preschoolers. *Developmental Psychology, 44*(2), 588-596. https://doi.org/10.1037/0012-1649.44.2.588

Wikfeldt, E. (2016). Generalising from case studies. Retrieved from https://www.diva-portal.org/smash/get/diva2:1051446/FULLTEXT01.pdf

Wilson, A. D., & Golonka, S. (2013). Embodied cognition is not what you think it is. *Frontiers in psychology*. https://doi.org/10.3389/fpsyg.2013.00058

Wilson, A. J., Dehaene, S., Dubois, O., & Fayol, M. (2009). Effects of an adaptive game intervention on accessing number sense in low-socioeconomic-status kindergarten children. *Mind, Brain, and Education, 3*(4), 224-234. https://doi.org/10.1111/j.1751-228X.2009.01075.x

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic bulletin & review, 9*(4), 625-636. https://doi.org/10.3758/BF03196322

Wilson, R. A., & Foglia, L. (2017). Embodied Cogntion. I E. N. Zalta (Red.), The Stanford Encyclopedia of Philosophy (Spring 2017 Edition) (utg., Bd., s.). Retrieved from https://plato.stanford.edu/archives/spr2017/entries/embodied-cognition

Winter, B., Matlock, T., Shaki, S., & Fischer, M. H. (2015). Mental number space in three dimensions. *Neuroscience & Biobehavioral Reviews, 57*, 209-219. https://doi.org/10.1016/j.neubiorev.2015.09.005

Wohlin, C. (2014). Guidelines for snowballing in systematic literature studies and a replication in software engineering. Proceedings of the 18th international conference on evaluation and assessment in software engineering- EASE '14, London.

Wolters, G., Van Kempen, H., & Wijlhuizen, G.-J. (1987). Quantification of small numbers of dots: Subitizing or pattern recognition? *The American journal of psychology*, 225-237. https://doi.org/10.2307/1422405

Wood, G., Willmes, K., Nuerk, H.-C., & Fischer, M. H. (2008). On the cognitive link between space and number: A meta-analysis of the SNARC effect. *Psychology Science, 50*(4), 489-525.

Wynn, K. (1990). Children's understanding of counting. *Cognition, 36*(2), 155-193. https://doi.org/10.1016/0010-0277(90)90003-3

Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology, 24*(2), 220-251. https://doi.org/10.1016/0010-0285(92)90008-P

Yin, R. K. (2009). *Case study research: Design and methods. Fourth edition*. Sage publications.

Young, A. G., & Shtulman, A. (2020). How Children's Cognitive Reflection Shapes Their Science Understanding. *Frontiers in psychology, 11*, 1247-1247. https://doi.org/10.3389/fpsyg.2020.01247

Yu, C. (2014). Linking words to world: An embodiment perspective. In *The Routledge Handbook of Embodied Cognition* (pp. 157-167). Routledge. https://doi.org/10.4324/9781315775845

Yun, C., Havard, A., Farran, D., Lipsey, M., Bilbrey, C., & Hofer, K. (2011). Subitizing and mathematics performance in early childhood. Proceedings of the Annual Meeting of the Cognitive Science Society,

Appendices

Appendix 1: Pilot study

A pilot study was conducted in an ECEC institution in the eastern part of Norway engaging four 4-year-olds' in a 6-week outdoor embodied design programme (Bjørnebye et al., 2017). The design of the 1- to 4-dotted arrays was framed within a square format and addressed the dual goal of using the arrays in both simultaneous and sequential physical mapping of numbers.


Figure 39: First step ("One") of the aerobic movement pattern


Figure 40: Second step ("Two") of the aerobic movement pattern


Figure 41: Third step ("Three") of the aerobic movement pattern


Figure 42: Keeping the balance in the fourth step ("Four")

The sequential approach reflected an adapted gait pattern known as the aerobic movement pattern. First, the right foot is moved forward followed by placing the left leg in parallel, and then the right leg is moved backward, followed by moving the left leg in parallel to complete a four-step cycle (see Figures 28-31). To support flow and rhythm, only one leg should touch the ground at a time. During the joint sessions, streamed music connected to mini-speakers provided focus for the children's engagement with the aerobic movement pattern (see the *Rhythmic tagging* activity in section 4.2.1).


Figure 43: Arrays used in the training sessions


Figure 44: Circle with 16 dots used in the modified Give-N post-task


Figure 45: High jump and bodily twist preceding the tagging

However, the main goal of the pilot study was to examine young children's abilities to use animal metaphors as verbal references in the physical production of small sets (Bjørnebye et al., 2017). The interventional experiences related to this part of the pilot-study involved simultaneously expressed physical couplings of the arrays in consecutive order while articulating a corresponding verbal referent (i.e., "rooster-one"/"cook-a-doodle-one", "kangaroo-two", "monkey-three" and "dog-four/cat-four/frog-four"; see Figure 43). Data from individual post-tests in a modified large scaled Give-N post-task (e.g., "Can you jump a monkey-three?") contextualised to a circle (d = 2 m; see Figure 44) with 16 arbitrarily distributed dots showed that the children enabled exact numbering in a manner that exceeded their assessed knower-level (cf. the standardised Give-N task). Additional lessons to be learned from the pilot study include the following observations:

1. The children needed guidance to discern numerosities in arrays and to use animal metaphors in verbal communication of their embodied action.
2. Some children attributed bodily creativity to the physical coupling of sets (e.g., making twists when entering into the arrays; see Figure 45).
3. Some children introduced novel metaphors (e.g., "Superman/Superman-two", "Bear-four", "Lightning McQueen-four", and "Horse-four").
4. Training contextualised to games and play improved motivation and consistency.
5. Sprayed arrays worked better than chalked ones due to rain, wear and tear.
6. The modified large-scale Give-N task proved to be an effective tool for outdoor measurement of children's abilities in the bodily production of small sets (see Figure 44).

To summarise, the pilot study provided valuable practical experience in implementing and conducting outdoor embodied design research in ECEC institutions, and it identified limitations, strengths, and new ways of engaging young children in physical learning of mathematics.

Appendix 2: NSD evaluation

NSD

Morten Bjørnebye
Avdeling for lærerutdanning og naturvitenskap - HiHM Høgskolen i Innlandet

2418 ELVERUM

Vår dato: 29.06.2017 Vår ref: 54530 / 3 / BGH Deres dato: Deres ref:

TILBAKEMELDING PÅ MELDING OM BEHANDLING AV PERSONOPPLYSNINGER

Vi viser til melding om behandling av personopplysninger, mottatt 25.05.2017. Meldingen gjelder prosjektet:

54530	*Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse: A multi-case embodied design study on early learning in mathematics: Perception, cognition and measurement of the cardinal concept in bodily-spatial interaction*
Behandlingsansvarlig	*Høgskolen i Innlandet, ved institusjonens øverste leder*
Daglig ansvarlig	*Morten Bjørnebye*

Personvernombudet har vurdert prosjektet og finner at behandlingen av personopplysninger er meldepliktig i henhold til personopplysningsloven § 31. Behandlingen tilfredsstiller kravene i personopplysningsloven.

Personvernombudets vurdering forutsetter at prosjektet gjennomføres i tråd med opplysningene gitt i meldeskjemaet, korrespondanse med ombudet, ombudets kommentarer samt personopplysningsloven og helseregisterloven med forskrifter. Behandlingen av personopplysninger kan settes i gang.

Det gjøres oppmerksom på at det skal gis ny melding dersom behandlingen endres i forhold til de opplysninger som ligger til grunn for personvernombudets vurdering. Endringsmeldinger gis via et eget skjema, http://www.nsd.uib.no/personvernombud/meld_prosjekt/meld_endringer.html. Det skal også gis melding etter tre år dersom prosjektet fortsatt pågår. Meldinger skal skje skriftlig til ombudet.

Personvernombudet har lagt ut opplysninger om prosjektet i en offentlig database, http://pvo.nsd.no/prosjekt.

Personvernombudet vil ved prosjektets avslutning, 01.08.2021, rette en henvendelse angående status for behandlingen av personopplysninger.

Vennlig hilsen

Kjersti Haugstvedt

Belinda Gloppen Helle

Kontaktperson: Belinda Gloppen Helle tlf: 55 58 28 74

Dokumentet er elektronisk produsert og godkjent ved NSDs rutiner for elektronisk godkjenning.

NSD – Norsk senter for forskningsdata AS Harald Hårfagres gate 29 Tel: +47-55 58 21 17 nsd@nsd.no Org.nr. 985 321 884
NSD – Norwegian Centre for Research Data NO-5007 Bergen, NORWAY Faks: +47-55 58 96 50 www.nsd.no

Appendix 3: Information letter with consent form – leaders of the ECEC institutions[23]

Til: NN
Leder ved NN barnehage

Høgskolen i Innlandet
Prosjektansvarlig: Morten Bjørnebye
Tlf: 91116376
Email: morten.bjornebye@inn.no Elverum, 23. mai 2017

Informasjon om forskningsprosjektet «Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse»

Våre navn er Morten Bjørnebye og Reinert Rinvold og vi arbeider til daglig ved Høgskolen i Innlandet. I tillegg til å forske på tidlig innlæring av matematiske begreper utdanner vi førskolelærere og lærere i matematikk ved campus Hamar. Reinert Rinvold er leder for en gruppe på fem ansatte ved Høgskolen i Innlandet som forsker på barnehagematetikk. Som en del av denne forskningen skal vi igangsette et nytt prosjekt knyttet til innlæring av matematiske begreper gjennom bevegelse og kroppslige uttrykk. Prosjektet «Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse» er således en del av et større forsknings- og utviklingsprosjekt ved Høgskolen i Innlandet, og det er fra Høgskolens side satt av midler til dette delprosjektet i en fire-års periode. I regi av sitt dr.grads arbeid er Morten Bjørnebye ansvarlig for dette prosjektet.

Nylig publiserte studier viser at norske barn sjeldent bruker tallord (en, to, tre, fire, …) til daglig. Sammenlignet med andre land er frekvensen langt lavere, og forskerne lurer på om det er kulturbetingede faktorer som spiller inn. Det er forskningsmessig belegg for at tidlig innsats gir læring, og dermed positive resultater på senere kartleggingsprøver i grunnskolen. Vi mener at barn gjennom bevegelsesaktiviteter som kobler tall og rytmer kan utvikle en helhetlig forståelse av tall på et tidlig stadium. Det å kunne uttrykke og behandle tall uten telling anses å være den mest avgjørende faktoren for at barna utvikler matematisk tallforståelse. Hensikten er å stimulere tidlig utvikling av tallbegrepet gjennom lek, bevegelse, rytme og samspill. Vi ønsker gjennom det å koble kroppslig rytme til utvikling av et meningsfullt tallbegrep, og som gjør at barnas spontane oppmerksomhet på tellbare strukturer i omgivelsene øker.

Gjennomføringen vil foregå innenfor barnehagens faste rammer med aktivitet inne og ute, men med spesiell vekt på å utnytte uterommet som læringsarena. Læringsøktene vil variere fra rutiner i barnehagehverdagen (f.eks. trampe av seg møkk på sko og klær i en firestegs-rytme før barnet går inn i barnehagen), til korte og intensive bevegelsesaktiviteter (f.eks. hoppe «kenguru-to» eller «froske-fire» i tallmatriser) og til lengre uteøkter (f.eks. undersøke hva som kan telles med «talldans» av naturmateriale). Lengden på det matematiske fokuset

[23] This is basically the same information letter provided to the participating KTs.

vil avhenge av barnas interesse for aktiviteten. Det er i hovedsak undertegnede i samarbeid med barnehagens personell som står ansvarlig for organisering og gjennomføring av aktiviteten. Det er ønskelig at de barna som deltar i studien følger et ukentlig opplegg i opp mot et og et halvt år med vekt på snøfrie måneder (perioden august til og med september, og fra april/mai til og med juni). Innhenting av informasjon om barnas samhandling og utvikling av tallforståelse vil skje gjennom observasjon og videoopptak av enkelte økter. Deltagere vil gjennomføre enkle kartlegginger i telling, tallbegrep og motorisk kompetanse.

Prosjektet er underlagt vanlige forskningsetiske retningslinjer og er således underlagt taushetsplikt. Det vil si at all informasjon vil bli anonymisert slik at det ikke er mulig å identifisere barn ved publikasjoner av forskningsresultat. Det søkes om tillatelse til forskningsprosjektet som ikke vil bli igangsatt før godkjenning er gitt av Personvernombudet for forskning, Norsk samfunnsvitenskapelig datatjeneste (NSD). Prosjektstart er planlagt til 1. august 2017 og planlagt prosjektslutt er 1. juli 2021. Innsamlet informasjon vil bli oppbevart og behandlet i henhold til NSD sine retningslinjer. Deler av videoopptakene vil bli brukt hvor forskerne presentere data fra undersøkelsen, men dette vil bli gjort på en konfidensiell måte.

Vi søker herved om at dere blir aktive partnere i gjennomføring av forskningsprosjektet «Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse».

Med vennlig hilsen

Morten Bjørnebye
Prosjektleder

Jeg bekrefter at jeg har fått muntlig og skriftlig informasjon om prosjektet og er innforstått med intensjonene. Jeg samtykker at prosjektet gjennomføres ved NN barnehage og at utvalgte barn deltar i prosjektet.

Sted/dato: ________________________

Barnehageleders underskrift: ____

Appendix 4: Information letter with consent form - parents

Til: Foresatte i NN

Høgskolen i Innlandet
Prosjektansvarlig: Morten Bjørnebye
Tlf: 91116376
Email: morten.bjornebye@inn.no

Elverum, 09. august 2017

Informasjon om forskningsprosjektet «Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse»

Våre navn er Morten Bjørnebye og Reinert Rinvold og vi arbeider til daglig ved Høgskolen i Innlandet. I tillegg til å forske på tidlig innlæring av matematiske begreper, utdanner vi førskolelærere og lærere i matematikk ved campus Hamar. Reinert Rinvold er leder for en gruppe på fem ansatte ved Høgskolen i Innlandet som forsker på barnehagematetikk. Som en del av denne forskningen skal vi igangsette et nytt prosjekt knyttet til innlæring av matematiske begreper gjennom bevegelse og kroppslige uttrykk. Prosjektet «Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse» er således en del av et større forsknings- og utviklingsprosjekt ved Høgskolen i Innlandet, og det er fra Høgskolens side satt av midler til dette delprosjektet i en fire-års periode. I regi av sitt dr.grads arbeid er Morten Bjørnebye ansvarlig for dette prosjektet.

Nylig publiserte studier viser at norske barn sjeldent bruker tallord (en, to, tre, fire, …) til daglig. Sammenlignet med andre land er frekvensen langt lavere, og forskerne lurer på om det er kulturbetingede faktorer som spiller inn. Det er forskningsmessig belegg for at tidlig innsats gir læring, og dermed positive resultater på senere kartleggingsprøver i grunnskolen. Vi mener at barn gjennom bevegelsesaktiviteter som kobler tall og rytmer kan utvikle en helhetlig forståelse av tall på et tidlig stadium. Det å kunne uttrykke og behandle tall uten telling anses å være den mest avgjørende faktoren for at barna utvikler matematisk tallforståelse. Hensikten er å stimulere tidlig utvikling av tallbegrepet gjennom lek, bevegelse, rytme og samspill. Vi ønsker gjennom det å koble kroppslig rytme til utvikling av et meningsfullt tallbegrep, og som gjør at barnas spontane oppmerksomhet på tellbare strukturer i omgivelsene øker.

Gjennomføringen vil foregå innenfor barnehagens faste rammer med aktivitet inne og ute, men med spesiell vekt på å utnytte uterommet som læringsarena. Læringsøktene vil variere fra rutiner i barnehagehverdagen (f.eks. trampe av seg møkk på sko og klær i en firestegs-rytme før barnet går inn i barnehagen), til korte og intensive bevegelsesaktiviteter (f.eks. hoppe «kenguru-to» eller «froske-fire» i tallmatriser) og til lengre uteøkter (f.eks. undersøke hva som kan telles med «talldans» av naturmateriale). Lengden på det matematiske fokuset vil avhenge av barnas interesse for aktiviteten. Det er i hovedsak undertegnede i samarbeid med barnehagens personell som står ansvarlig for organisering og gjennomføring av aktiviteten. Det er ønskelig at de barna som deltar i studien følger et ukentlig opplegg i opp mot et og et halvt år med vekt på snøfrie måneder (perioden august til og med september, og fra april/mai til og med juni). Innhenting av informasjon om barnas samhandling og utvikling av tallforståelse vil skje gjennom observasjon og videoopptak av enkelte økter. Deltagere vil gjennomføre enkle kartlegginger i telling, tallbegrep og motorisk kompetanse.

Prosjektet er underlagt vanlige forskningsetiske retningslinjer og er således underlagt taushetsplikt. Det vil si at all informasjon vil bli anonymisert slik at det ikke er mulig å identifisere barn ved publikasjoner av forskningsresultat. Prosjektet er meldt til Personvernombudet for forskning, NSD - Norsk senter for forskningsdata AS. Prosjektstart er planlagt til 1. august 2017 og planlagt prosjektslutt er 1. juli 2021. Innsamlet informasjon vil bli oppbevart og behandlet i henhold til NSD sine retningslinjer. Deler av videoopptakene vil bli brukt hvor forskerne presentere data fra undersøkelsen, men dette vil bli gjort på en konfidensiell måte. Datamaterialet skal anonymiseres ved prosjektslutt.

Med vennlig hilsen

Morten Bjørnebye
Prosjektleder

Samtykkeerklæring til forskningsprosjektet «Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse»

Jeg/vi har fått skriftlig informasjon om forskningsprosjektet «Tidlig innlæring av tallbegrepet gjennom kropp og bevegelse» i NN, og er innforstått med hva mitt barn deltagelse i prosjektet innebærer. Jeg/vi samtykker at mitt barn deltar i prosjektet. Jeg/vi er innforstått med at medvirkning i studien er frivillig og at jeg/vi når som helst kan velge å avbryte mitt barns deltagelse i studien uten å begrunne hvorfor.

Sted/dato: ________________________ Barnets navn: _______________________________

Foresattes underskrift: ____

Appendix 5: Author's declarations

Co-author Statement

In this statement, the PhD candidate and all article co-authors must provide information about their individual contributions to the article indicated below. Co-author statements must be provided for every article with multiple authors included in the thesis. The authors hereby confirm that they understand that this article has been submitted as part of a doctoral thesis. They also confirm that the article has not previously been submitted as part of a doctoral thesis, or in another way been used to qualify for a PhD degree, in Norway or abroad. This statement must follow the recommendations of The International Committee of Medical Journal Editors, the so-called Vancouver Recommendations. Co-authorship should be based on the following criteria:

1. Substantial contributions to the conception or design of the work; or the acquisition, analysis, or interpretation of data for the work; AND
2. Drafting the work or revising it critically for important intellectual content; AND
3. Final approval of the version to be published; AND
4. Agreement to be accountable for all aspects of the work in ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.

Both the PhD candidate and the co-author(s) are required to report on their contributions in each article, in addition to signing for each of the respective articles. All authors must sign on the last page to confirm that they have read the entire statement and that the information provided is correct. More than one form may be used per article if necessary.

The form must be signed and attached to the Application for the Assessment of the Thesis for the degree philosophiae doctor at Inland Norway University of Applied Sciences.

For more information about authorship see:

http://www.icmje.org/icmje-recommendations.pdf

Article Number:	1
Title	Bjørnebye, M., & Sigurjonsson, T. (2020). Young Children's Cross-Domain Mapping of Numerosity in Path Navigation. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics Education in the Early Years: Results from the POEM4 Conference, 2018* (pp. 109-126). Cham: Springer International Publishing.
PhD Candidate's Name	Morten Bjørnebye
Authors (in the same order as they appear on the article)	Morten Bjørnebye og Thorsteinn Sigurjonsson

Candidate's Contribution:

- I contributed to the choice of the theory used in the article and to the formulation of the scientific problem.
- I contributed to the design and the implementation of the embodied training program 1.
- I did majority of preparing the data collection, conducting the intervention, evaluating and re-designing the program, creating the testing procedure and collecting the empirical data.
- I did the majority of work when transcribing, coding, analysing and selecting empirical material for the multi-case analysis.
- I did the majority of work in writing the first draft of the article
- I presented an early version of the article at the POEM 4 conference in Kristiansand.
- I contributed to writing the final version of the article.

Co-author's name:	Thorsteinn Sigurjonsson

- He contributed to the choice of the theory used in the article and to the formulation of the scientific problem.
- He contributed to the design of the embodied training program 1, in evaluation and re-design of the program.
- He contributed to preparing the data collection and creating the testing procedure.
- He contributed to coding, analysing and selecting empirical material for the multi-case analysis.
- He provided feedback on written versions of the article and contributed to sections of the text and to proofreading.

I have read the authors' statements and confirm that all of the information is correct.

Morten Bjørnebye

Elverum, 10.02.2021

Candidate's signature

Thorsteinn Sigurjonsson

Elverum, 10.02.2021

Co-author's signature

Article Number:	2
Title	Bjørnebye, M., & Sigurjonsson, T. (Unpublished). Young children's simulated action in additive reasoning
PhD Candidate's Name	Morten Bjørnebye
Authors (in the same order as they appear on the article)	Morten Bjørnebye og Thorsteinn Sigurjonsson

Candidate's Contribution:

- I contributed to the choice of the theory used in the article and to the formulation of the scientific problem.
- I contributed to the design and the implementation of the embodied training program 1.
- I did the majority of preparing the data collection, conducting the intervention, evaluating and re-designing the program, creating the testing procedure and collecting the empirical data.
- I did the majority of work when transcribing, coding, analysing and selecting empirical material for the multi-case analysis.
- I did the majority of work in writing the first version of the article
- I presented the study for the field of practice.
- I contributed to writing the final version of the article.

Co-author's name:	Thorsteinn Sigurjonsson

- He contributed to the choice of the theory used in the article and to the formulation of the scientific problem.
- He contributed to the design of the embodied training program 1, in evaluation and re-design of the program.
- He contributed to preparing the data collection and creating the testing procedure.
- He contributed to coding, analysing and selecting empirical material for the multi-case analysis.
- He provided feedback on written versions of the article and he contributed to sections of the text and to proofreading.

I have read the authors' statements and confirm that all of the information is correct.

Morten Bjørnebye

-- Elverum, 10.02.2021

Candidate's signature

Thorsteinn Sigurjonsson

-- Elverum, 10.02.2021

Co-author's signature

Inland Norway
University of
Applied Sciences

In this thesis, I explore characteristics of 3- to 5-year-olds' grounding of mathematical thinking in sensory-motor experiences. The study reports from design-based interventions conducted outdoors in two kindergartens in Norway. The study contributes to the field of educational research on early structure-based bodily learning of mathematics, and it develops understanding of how embodied designs can facilitate such experiences.

With children unable to master verbal counting as a target group, the first focus study examined their production of small sets through speech and bodily interaction in a circle with 50 dots. The second focus study examined the children's abilities to re-enact symmetrically structured bodily experiences with numbers to support additive reasoning, while the third focus study explored coherence in speech and bodily modelling of counting-based addition.

In light of theory of Embodied Cognition, the results showed patterns of the children's bodily production of small sets that also exceeded their measured concept level (cf. standardised tests), and the findings showed how sensory-motor action might concur with counting-based addition and support reasoning about additive compositions. Unexpected findings were the integration of composite and expressive body movements (e.g., rotation, rhythm, force, and tempo) in the physical grounding of mathematical thinking.

The results should encourage the design of outdoor activities that involve movement and rhythm in the early learning of mathematics. The study shows that embodied designs should be considered a suitable approach for realising some of the mathematical targeting goals of the *Norwegian Framework Plan for Kindergartens*.